Stable methods for solving the standard Abel integral equation by means of orthogonal polynomials.

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Outline

Introduction

The standard form of the Abel integral equation is given as follows,

$$T_\alpha u(x) = \int_0^x \frac{u(t)}{(x-t)^{1-\alpha}} dt = f(x), \quad 0 \leq x \leq 1.$$

where $0 < \alpha < 1$ is a positive real number, $f(\cdot)$ is the data function and $u(\cdot)$ is the unknown function to be computed.
In this talk, we built **stable methods** for the solution of the **ill-posed problem**.
These methods are **explicit** and they are **based on the use of various families of orthogonal polynomials** of the Legendre and Jacobi types.
Moreover, they have the **advantage to ensure** the stability of the solution under a fairly weak condition on the functional spaces to which the data function belongs. Also, we provide some numerical examples that illustrate our proposed methods.

Theme 2 : Stable solution of Abel integral equation by means of orthogonal polynomials

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- The first method is based on the use of Jacobi-types orthogonal polynomials that constitute a basis of $L^2([0,1], (1-x)^\alpha x^\beta dx)$. More precisely, for real numbers $\alpha, \beta > -1$, we consider the set of orthonormal polynomials

$$Q_n^{\alpha,\beta}(x) = \sqrt{\frac{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}} P_n^{\alpha,\beta}(2x-1), \qquad (2)$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the $n-$th degree Jacobi polynomial.

TWO STABLE METHODS FOR SOLVING THE STANDARD ABEL INTEGRAL EQUATION

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- $P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}((1-x)^{n+\alpha}(1+x)^{n+\beta}).$

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$$T_\alpha(w^\beta Q_n^{\mu,\beta})(w) = \Gamma(\alpha)\sqrt{\frac{\Gamma(n+\mu-\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\beta+\alpha+1)\Gamma(n+\mu+1)}}\, w^{\beta+\alpha} Q_n^{\mu-\alpha,\beta+\alpha}(w).$$

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- We know that if $u,\ f$ in $L^2([0, 1], dx)$, then the solution of the Abel equation $T_\alpha u = f$ satisfies the following identity $\int_0^x u(t)dt = \frac{\sin(\alpha\pi)}{\pi} T_{1-\alpha} f(x),\ 0 \leq x \leq 1.$

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- Let E_α denote the normed space given as follows,

$$E_\alpha = \left\{ f = \sum_{n=0}^{\infty} f_n Q_n^{1-\alpha,0} \in L^2([0, 1], (1-x)^{1-\alpha} dx);\ \sum_{n\ge 0} |f_n|\, n^{\alpha+\frac{3}{2}} < \infty \right\}.$$

A norm $\|\cdot\|_\alpha$ is defined on E_α by, $\|f\|_\alpha^2 = \sum_{n\ge 0} f_n^2 (n+2)^{2\alpha}$.

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Proposition

Let $0 < \alpha < 1$, *and assume that* $f = \sum_{n=0}^{\infty} f_n Q_n^{1-\alpha,0} \in E_\alpha$, *then the solution of the equation* $T_\alpha u = f$ *is given by the following formula :*

$$\begin{aligned} u(x) &= \frac{\sin(\alpha\pi)}{(1-\alpha)\pi} \sum_{n\geq 0} f_n \frac{(-1)^n \sqrt{(2n+2-\alpha)}}{(2-\alpha)_n} h_n(x) \\ &= \frac{\sin(\alpha\pi)}{(1-\alpha)\pi} \sum_{n\geq 0} f_n \frac{(-1)^n \Gamma(1-\alpha) \sqrt{(2n+2-\alpha)}}{\Gamma(n+2-\alpha)} h_n(x), \quad (3) \end{aligned}$$

where $h_n(x) = \frac{d^{n+1}}{dx^{n+1}}\left((1-x)^n x^{n+1-\alpha}\right), \quad x \in [0,1].$

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Theorem

Let $0 < \alpha < 1$ and assume that $f \in E_\alpha$, where the space E_α is given by the previous formula. Then, the solution u of the equation $T_\alpha u = f$ belongs to the weighted space $L^2([0,1], d\sigma) = L^2([0,1], (1-x)x^\alpha\, dx)$. Moreover, this **solution is stable in the sense** *that if $\widetilde{f} = \sum_{n=0}^{\infty} \widetilde{f}_n Q_n^{1-\alpha,0} \in E_\alpha$ and $\widetilde{u} \in L^2([0,1], d\sigma)$ satisfy $T_\alpha \widetilde{u} = \widetilde{f}$, then we have*

$$\|u - \widetilde{u}\|_\sigma \leq \frac{1}{\Gamma(\alpha)} \left\| f - \widetilde{f} \right\|_\alpha, \tag{4}$$

where $\|\cdot\|_\sigma$ is the usual norm of $L^2([0,1], d\sigma)$.

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Proposition

Let $N \geq 1$ be a positive integer and let

$$\widetilde{u}_N(x) = \frac{\sin(\alpha\pi)}{\pi}\sum_{n=0}^{N-1}\widetilde{f}_n\frac{(-1)^n\Gamma(1-\alpha)\sqrt{(2n+2-\alpha)}}{\Gamma(n+2-\alpha)}h_n(x),$$

be the approximate solution of equation $T_\alpha\widetilde{u} = \widetilde{f}$. Here, the $\left(\widetilde{f}_n\right)_{0\leq n\leq N-1}$ are the expansion coefficients of the perturbed function $\widetilde{f}$. Assume that $\|f-\widetilde{f}\|_\alpha \leq \Gamma(\alpha)\varepsilon$, where f is the noise free function, satisfying $T_\alpha u = f$. Then, we have

$$\|u-\widetilde{u}_N\|_\sigma \leq \frac{1}{\Gamma(\alpha)}\sqrt{\sum_{n=N}^{\infty}f_n^2(n+2)^{2\alpha}} + \varepsilon$$

Remark : In practice, for smooth enough function f, the remainder term $\sqrt{\sum_{n=N}^{\infty} f_n^2 (n+2)^{2\alpha}}$ has a fast decay. In fact under the **assumption that there exist** a positive real number $s > \alpha + 1/2$ **and a constant** $k > 0$ such that $|f_n| \leq k(n+2)^{-s}, \forall\, n \geq 1$, we have $\sum_{n \geq N} f_n^2 (n+2)^{2\alpha} \leq k^2 \sum_{n \geq N+2} n^{2(\alpha - s)} \leq k^2 \int_{N+1}^{+\infty} x^{2(\alpha - s)}\, dx = \frac{k^2}{(2(\alpha - s)+1)(N+1)^{2(s-\alpha)-1}}$. The appropriate number of modes N_ε, to be used is given in this case is given by,

$$N_\varepsilon = \inf \left\{ N \geq 1,\ \frac{k^2}{2(\alpha - s)+1} \frac{1}{(N+1)^{2(s-\alpha)-1}} \leq \| f - \widetilde{f} \|_\alpha^2 \right\}.$$

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Let $0 < \alpha < 1$ *and let* $f = \sum_{s\geq 0} f_s P_s(t) \in L^2([0,1])$. *If* $u(t) = \sum_{m\geq 0} \alpha_m P_m(t) \in L^2([0,1])$ *is the solution of the Abel integral equation* $T_\alpha u = f$, *then* $\forall\, m \geq 0$, *we have* $\alpha_m =$

$$\sum_{s\geq 0} f_s (-1)^s \frac{\sqrt{2m+1}\sqrt{2s+1}}{m!\Gamma(\alpha)} \beta_{m,\alpha} \sum_{l=0}^{m} (-1)^l \binom{m}{l} \prod_{j=0}^{s-1} (l-\alpha-j) \frac{\Gamma(m+l+1)}{\Gamma(l-\alpha+s+2)}.$$

where $\beta_{m,\alpha} = \Gamma(\alpha)\sqrt{\frac{\Gamma(m-\alpha+1)}{\Gamma(m+\alpha+1)}} \sim m^{-\alpha}$

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where $\beta_{m,\alpha} = \Gamma(\alpha)\sqrt{\frac{\Gamma(m-\alpha+1)}{\Gamma(m+\alpha+1)}} \sim m^{-\alpha}$

- Let $\sigma_0(\alpha) > 0$, be the positive real number, given as follows,

$$\sigma_0(\alpha) = \inf\left\{\eta > 0, \sum_{m\geq 0}\sum_{s\geq 0} \frac{< P_s, F_m >^2}{\beta^2_{m,\alpha}(1+(s(s+1))^\eta)^2} < +\infty\right\}. \quad (5)$$

where $F_m(x) = (1-x)^{-\alpha} Q_m^{-\alpha,\alpha}(x)$.

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- For any real number $\sigma > 0$, we defined the space

$$D(\mathcal{A}^\sigma) = \left\{ u = \sum_{n\geq 0} \widehat{u}_n P_n,\ \sum_{n\geq 0}(1 + (n(n+1))^\sigma)^2 |\widehat{u}_n|^2 < +\infty \right\},$$

equipped by a norm $\|\cdot\|_\sigma$ given by
$\|u\|_\sigma^2 = \sum_{n\geq 0}(1 + (n(n+1))^\sigma)^2 |\widehat{u}_n|^2$. And
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- Under the above notation, the following proposition shows that our second method for solving the Abel equation (1) is stable.

Proposition

Let $0 < \alpha < 1$ and let σ be any positive real number satisfying $\sigma > \sigma_0(\alpha)$. Assume that $f \in D(\mathcal{A}^\sigma)$, then **the solution of** *the equation $T_\alpha u = f$,* **given by the previous method is stable**.

Corollary

Let $0 < \alpha < 1$, *let* $\sigma > \sigma_0(\alpha)$ *be a positive real number and let* $N \geq 0$, *be a positive integer. Assume that* $f \in D(\mathcal{A}^\sigma)$ *and let* $\widetilde{f}$, *be the perturbation of the function* f, *given by* $\widetilde{f} = f + \eta$, *where* $\eta \in L^2([0,1])$, *is the noise function. If* $u, \widetilde{u}_N$, *denote the solutions of* $T_\alpha u = f$, $T_\alpha \widetilde{u}_N = (\pi_N \widetilde{f})$, *then we have* $\forall 0 < \epsilon < \sigma - \sigma_0(\alpha)$.
$\|u - \widetilde{u}_N\|_{L^2([0,1])} \leq \frac{\sqrt{C_\alpha}\|f\|_\sigma}{((N+1)(N+2))^{\sigma-\sigma_0(\alpha)-\epsilon}} + \sqrt{C_\alpha}\|\pi_N \eta\|_{\sigma_0(\alpha)+\epsilon}$.

Corollary

Let $0 < \alpha < 1$, let $\sigma > \sigma_0(\alpha)$ be a positive real number and let $N \geq 0$, be a positive integer. Assume that $f \in D(\mathcal{A}^\sigma)$ and let $\widetilde{f}$, be the perturbation of the function f, given by $\widetilde{f} = f + \eta$, where $\eta \in L^2([0,1])$, is the noise function. If $u, \widetilde{u}_N$, denote the solutions of $T_\alpha u = f$, $T_\alpha \widetilde{u}_N = (\pi_N \widetilde{f})$, then we have $\forall 0 < \epsilon < \sigma - \sigma_0(\alpha)$.

$$\|u - \widetilde{u}_N\|_{L^2([0,1])} \leq \frac{\sqrt{C_\alpha}\|f\|_\sigma}{((N+1)(N+2))^{\sigma - \sigma_0(\alpha) - \epsilon}} + \sqrt{C_\alpha}\|\pi_N \eta\|_{\sigma_0(\alpha)+\epsilon}.$$

- **Remark :** Numerical evidences show that $\sum_{s\geq 0} \frac{<P_s, F_m>^2}{(1+(s(s+1))^\eta)^2} = O\left(m^{-4\eta}\right)$. Moreover, since $\beta^2_{m,\alpha} \sim m^{-2\alpha}$, then we can chose $\sigma_0(\alpha) = \frac{1}{4} + \frac{\alpha}{2}$ defined by (5). This means that our Legendre based method for solving (1) is numerically valid on a fairly large subspace $\widetilde{E}_{\alpha,\epsilon}$ of $L^2([0,1])$, given by

$$\widetilde{E}_{\alpha,\epsilon} = \left\{ f = \sum_{n=0}^{\infty} f_n P_n \in L^2([0,1]);\ \sum_{n\geq 0} (1 + (n(n+1))^{1/4+\alpha/2+\epsilon})^2 (f_n)^2 < \infty \right\}$$

EXAMPLE 1 : SOLVED BY THE FIRST METHOD

We first consider $\alpha = 1/10$ and we apply our proposed method with $N = 25$ to solve equation (1), with the noisy data function $\widetilde{f}_1(x) = f_1(x) + \eta(x)$, where $f_1(x) = 10(x - \frac{1}{4})^{0.9} 1_{[\frac{1}{4},1]}(x)$, $\eta(x) = 0.01\sin(100\pi x)$, $0 \leq x \leq 1$.

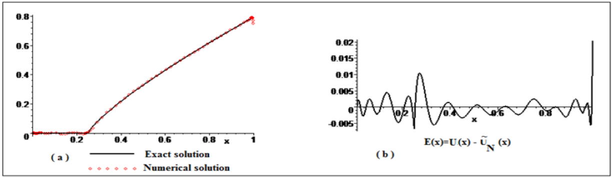

Figure: (a) graphs of the exact and numerical solution, (b) graph of the error corresponding to the function $\widetilde{f}_1$.

Example 2 : solved by the second method

In this example, we consider the value $\alpha = 0.5$ and the smooth function f_2, given by $f_2(x) = \frac{16}{3\Gamma(0.5)}(x^{5/2} - x^{3/2})$. In this case, the exact solution of (1) is given by $u(x) = 5x^2 - 4x$. Then, we contaminate the function f_2 with a **35 db SNR Gaussian** white noise η_2. Here, the SNR denotes the signal to noise ratio, given by the formula $\text{SNR} = 20\log_{10}\left(\frac{\|f\|_2}{\|\eta\|_2}\right)$, where f is the signal function and η is the noise. The graph of noisy function $\widetilde{f}_2$, is given by Figure 2(a). Next, we apply the Legendre polynomials based method with $N = 8$ and compute an approximate solution $\widetilde{u}_N(x)$ to the exact solution $u(x)$. The graphs of $u(x)$ and $\widetilde{u}_N$ are given by Figure 2(b).

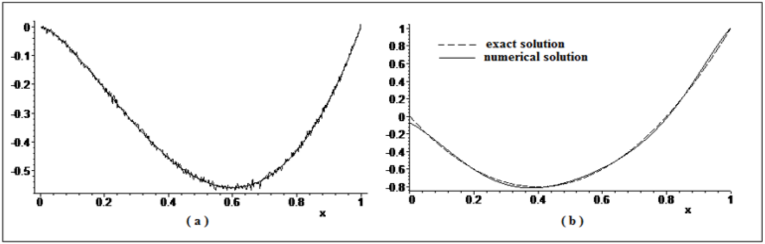

Figure: (a) graph of the noised function $\widetilde{f}_2$, (b) graphs of the exact and numerical solution corresponding to $\widetilde{f}_2$.

Theme 3 : Another Stable method for solving the Abel deconvolution problem

$$T_\alpha u(x) = \int_0^x \frac{u(t)}{(x-t)^{1-\alpha}} dt = f(x) = \sum_{n=0}^{\infty} f_n Q_n^{\alpha,\beta}$$

- We use a Jacobi-Legendre based method for the stable solution of the problem (1). This method is valid under the weak condition that $f(\cdot)$ belongs to the following functional space

- We use a Jacobi-Legendre based method for the stable solution of the problem (1). This method is valid under the weak condition that $f(\cdot)$ belongs to the following functional space
- $$F_\alpha = \left\{ \begin{array}{c} f = \sum_{n=0}^{\infty} f_n Q_n^{1-\alpha,0} \in L^2([0,1],(1-x)^{1-\alpha}dx); \\ \|f\| = \sum_{n\geq 0} |f_n| \sqrt{2n+2-\alpha} < \infty \end{array} \right\}$$

- We use a Jacobi-Legendre based method for the stable solution of the problem (1). This method is valid under the weak condition that $f(\cdot)$ belongs to the following functional space
- $$F_\alpha = \left\{ \begin{array}{c} f = \sum_{n=0}^{\infty} f_n Q_n^{1-\alpha,0} \in L^2([0,1],(1-x)^{1-\alpha}dx); \\ \|f\| = \sum_{n\geq 0} |f_n| \sqrt{2n+2-\alpha} < \infty \end{array} \right\}$$
- For a given three real numbers α, β, μ satisfying $0 < \alpha < 1,\ 2\alpha > \beta > -1,\ \mu > 0,$ we consider the weighted Sobolev type functional space $F_{\alpha,\beta,\mu}$, given by

$$F_{\alpha,\beta,\mu} = \left\{ \begin{array}{c} f = \sum_{n\geq 0} f_n x^{\alpha+\beta} Q_n^{\mu-\alpha,\beta+\alpha} \in L^2([0,1], x^{-\alpha-\beta}(1-x)^{\mu-\alpha}dx) \\ ;\ \sum_{n\geq 0}(f_n)^2(1+n)^{2\alpha} < +\infty \end{array} \right\}$$

Then, we prove that the Abel equation (1) has a stable solution belonging to $L^2([0,1],(1-x)^{\mu}x^{-\beta}dx)$, whenever $f \in F_{\alpha,\beta,\mu}$.

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Theorem

For any positive integers n, k *and any real* $0 < \alpha < 1$, *we have*

$$\begin{aligned} I_{n,k}(\alpha) &= \sqrt{2n+2-\alpha}\sqrt{2k+1} - \int_0^1 x^{1-\alpha} Q_n^{0,1-\alpha}(x) P_k'(x) dx \\ &= \frac{\sin(\pi\alpha)}{\pi} \frac{(-1)^{n+k}\sqrt{2n+2-\alpha}\sqrt{2k+1}}{n!} \\ & \sum_{l=0}^{n} \binom{n}{l} \frac{\Gamma(n+l+2-\alpha)\Gamma(l-\alpha+1)\Gamma(k-l+\alpha)}{\Gamma(k+l+2-\alpha)} \quad (6) \end{aligned}$$

Explicit solution of (1)

Theorem

Let $0 < \alpha < 1$, and assume that $f \in F_\alpha$. If $u(t) = \sum_{k\geq 0} \beta_k P_k(t)$ is the solution of the equation $T_\alpha u = f$, then for all $k \geq 1$ β_k is given by the following formula

$$\beta_k = \sqrt{2k+1}\beta_0 - \frac{\sin(\alpha\pi)}{\pi}\int_0^1 T_{1-\alpha} f(x) P_k'(x)dx$$

$$= \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{k-1} f_n \frac{n!}{\Gamma(n+2-\alpha)} I_{n,k}(\alpha) \quad (7)$$

$$+\frac{\sqrt{2k+1}}{\Gamma(\alpha)}\sum_{n=k}^{\infty} f_n \frac{n!}{\Gamma(n+2-\alpha)}\sqrt{2n+2-\alpha}, \quad (8)$$

where $I_{n,k}(\alpha)$ is as given by previous theorem.

Stability of the proposed solution of the Abel equation

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Theorem

For any positive integers n, k and any real number $0 < \alpha < 1$, *we have*

$$|I_{n,k}(\alpha)| \leq \frac{\sqrt{2n+2-\alpha}\,\Gamma(n+2-\alpha)}{n!}\chi_k(\alpha), \tag{9}$$

with

$$\chi_k(\alpha) = \frac{\sqrt{2k+1}k^\alpha}{(k+1-\alpha)}\left(2^\alpha - 1 + 2^\alpha \frac{\alpha}{2k}\right) \leq 2^{1/2+\alpha}\left(1+\frac{\alpha}{2-\alpha}\right)k^{\alpha-\frac{1}{2}}.$$

Theorem

Let $0 < \alpha < 1$, *be a real number and let* U_α *be the subspace of* $L^2([0,1])$, *given by*

$$U_\alpha = \left\{ u \in L^2([0,1]) = \sum_{k\geq 0} \beta_k P_k;\ \|u\|_\alpha = \sup_{k\geq 0} \left(|\beta_k| (k+1)^{\frac{1}{2}-\alpha} \right) < +\infty \right\}$$

Let $f = \sum_{n=0}^{\infty} f_n Q_n^{1-\alpha,0} \in F_\alpha, \widetilde{f} = \sum_{n=0}^{\infty} \widetilde{f}_n Q_n^{1-\alpha,0} \in F_\alpha$ *and let* $u = \sum_{k\geq 0} \beta_k P_k(x) \in U_\alpha, \widetilde{u} = \sum_{k\geq 0} \widetilde{\beta_k} P_k(x) \in U_\alpha$ *be the solutions of the equation* $T_\alpha u = f$, *and* $T_\alpha \widetilde{u} = \widetilde{f}$ *respectively. Then, we have*

$$\|u - \widetilde{u}\|_\alpha \quad \leq \quad C_\alpha \left\| f - \widetilde{f} \right\|.$$

with $C_\alpha = \frac{1}{\Gamma(\alpha)} \max \left(\frac{1}{(1-\alpha)}, 2^{2+\alpha}(1 + \frac{\alpha}{2-\alpha}) \right)$

Remark :

- In practice, for a given integer $N \geq 1$, we only compute $\widetilde{u}_N(x)$, a truncated version of $\widetilde{u}(x)$, given by the following formula,

$$\widetilde{u}_N(x) = \sum_{k=0}^{N-1} \widetilde{\beta}_k P_k(x), \quad 0 \leq x \leq 1, \tag{10}$$

where the $\widetilde{\beta}_k$ are as given by Theorem 2.

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- Note that if $\|f - \widetilde{f}\| = \epsilon$, and $\sup_{n \geq N+1} |\beta_n|(n+1/2)^{1/2-\alpha} \underset{N \mapsto \infty}{\longrightarrow} 0$ then by using the previous theorem, one obtains the following bound of the approximation error,

$$\|u-\widetilde{u}_N(x)\|_\alpha \leq \|u-u_N\|_\alpha + \|u_N-\widetilde{u}_N\|_\alpha \leq \sup_{n \geq N+1} |\beta_n|(n+1/2)^{1/2-\alpha} + C_\alpha \epsilon. \tag{11}$$

Second method

A second but a more classical stability result concerning the solution of the Abel integral equation (1) based on the use of Jacobi polynomials is given by the following theorem.

Theorem

Let α, β, μ, be any real numbers, satisfying $0 < \alpha < 1$, $\mu > 0$, and $2\alpha > \beta > -1$. If $f(\cdot) \in F_{\alpha,\beta,\mu}$, then the function

$$u(x) = \sum_{n\geq 0} f_n \eta_{n,\beta,\mu}(\alpha) x^{\beta} Q_n^{\mu,\beta}(x), \tag{12}$$

$$\eta_{n,\beta,\mu}(\alpha) = \frac{1}{\Gamma(\alpha)} \sqrt{\frac{\Gamma(n+\beta+\alpha+1)\Gamma(n+\mu+1)}{\Gamma(n+\mu-\alpha+1)\Gamma(n+\beta+1)}} \tag{13}$$

belongs to $L^2([0,1],(1-x)^{\mu}x^{-\beta}dx)$ and it is a solution of (1). Moreover, we have

$$\|u\|^2 \leq \frac{(1+\mu)^{\alpha}(1+\beta)^{\alpha}}{\Gamma^2(\alpha)} \sum_{n\geq 0} |f_n|^2 (1+n)^{2\alpha} = \frac{(1+\mu)^{\alpha}(1+\beta)^{\alpha}}{\Gamma^2(\alpha)} \|f\|^2. \tag{14}$$

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- By the work of **R. Gorenflo and M. Yamamato [4]**, we can obtain similar stability results associated with the solution of the generalized problem of Abel type by Jacobi type polynomials.

$$\mathcal{K}u(x) = \int_0^x \frac{k(x,t)}{(x-t)^{1-\alpha}} u(t)dt = f(x), \; 0 \leq x \leq 1.$$

Here, $k(\cdot)$ is a smooth function with $k(0) \neq 0$.

Example :

This example is devoted to the *challenging problem* of solving an Abel integral equation having a discontinuous solution have been studied. We consider the data function $f(x)$, given as follows,

$$f(x) = \begin{cases} 2\sqrt{x} & \textit{if } 0 \le x < 0.2 \\ 2\sqrt{x} - \frac{2}{3}(x-0.2)^{3/2} + (2x-4)\sqrt{x-0.2} = f_1(x) & \textit{if } 0.2 \le x < 0.5 \\ f_1(x) + \frac{4}{3}(x-0.5)^{3/2} - (4x-4)\sqrt{x-0.5} = f_2(x) & \textit{if } 0.5 \le x < 0.7 \\ f_2(x) - \frac{2}{3}(x-0.7)^{3/2} + (2x-3)\sqrt{x-0.7} & \textit{if } 0.7 \le x \le 1. \end{cases}$$

The exact solution of (1) corresponding to the previous data function is given by the following discontinuous function

$$u(x) = 1 \cdot 1_{[0,0.2[} + (1-x) \cdot 1_{[0.2,0.5[} + (x-1) \cdot 1_{[0.5,0.7[} - \frac{1}{2} 1_{[0.7,1]}, \;\; 0 \le x \le 1$$

We contaminated $f(x)$ *by a* **30 db SNR Gaussian** *white noise* $\eta(x)$. Next, we have computed the Jacobi expansion coefficients of $\widetilde{f}(x)$ by using a $N_q = 50$ points quadrature method, associated with the orthogonal polynomial $Q_{N_q}^{1-\alpha,0}(x), \alpha = \frac{1}{2}$. Then, we have applied formula (10) with $N = 40$ and obtained $\widetilde{u}_{40}(x)$, a numerical approximation to $u(x)$.

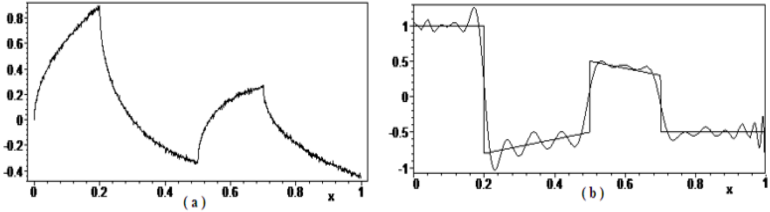

Figure: (a) Graphs of the functions $f(x)$ and the noised function $\widetilde{f}(x)$, (b) Graphs of the exact and the numerical solution $u(x), \widetilde{u}_{40}(x)$.

Remark the existence of a **Gibs phenomenon** at these discontinuity points $x_1 = 0.2, x_2 = 0.5, x_3 = 0.7$. To overcome the loss of accuracy problem and as it is done **in Ammari-Karoui, J. Cheng ([1],[3])**, we defined the function $F(x)$ obtained from $f(x)$ by using the following formula,

$$F(x) = f(x) \cdot 1_{[0,x_1]} + \sum_{p=1}^{3} \left[f(x) - \left(\sum_{l=1}^{p} \sigma_l \right) \int_{x_p}^{x} \frac{1}{(x-t)^{1-\alpha}} \right] 1_{]x_p, x_{p+1}]}, \; x_4 = 1. \tag{15}$$

$\sigma_l = u(x_l^+) - u(x_l^-), l = 1, \ldots, 3$ denotes the values of the jumps at the discontinuity point x_l. *If* $U(x)$ **denotes the solution** *of the Abel equation corresponding to the* **data function** $F(x)$, then it can be easily checked that the exact solution of the original problem is given as follows,

$$u(x) = U(x) \cdot 1_{[0,x_1]} + \sum_{p=1}^{3} \left[U(x) + \sum_{l=1}^{p} \sigma_l \right] 1_{]x_p, x_{p+1}]}, x_4 = 1. \tag{16}$$

By using the numerical estimates values $x_1^* = 0.206,\ x_2^* = 0.496,\ x_2^* = 0.706., \widetilde{\sigma}_1 = -1.81,\ \widetilde{\sigma}_2 = 0.96,\ \widetilde{\sigma}_3 = -0.89.$ we have computed **via formula** (15) a smooth function $\widetilde{F}(x)$. Then, we have resolved the Abel equation by using our Jacobi-Legendre method with $N = 10$ and the auxiliary data function $\widetilde{F}(x)$. Finally, we have applied formula (16) and obtained a highly accurate approximation $\widetilde{u}_{10}(x)$ to the true solution $u(x)$, see Figure 5.

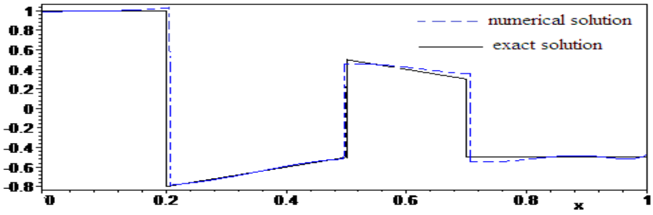

Figure: Graphs of the exact and the numerical solution $u(x), \widetilde{u}_{10}(x)$.

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Thank You