Uncertainty Principle and Prolate Spheroidal Wave functions

Aline Bonami,
Université d’Orléans

Joint work with Abderrazek Karoui

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Heisenberg Uncertainty Principle

Heisenberg Inequality (in Dimension one) is

$$\Delta x \Delta p \geq \hbar / 2 \quad \hbar = h / 2\pi.$$
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Mathematically, this is equivalent to the inequality

$$\left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right)^{1/2} \times \left( \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \geq \int_{\mathbb{R}} |f(x)|^2 \, dx,$$

with equality for Gaussian functions.

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} f(x) \, dx.$$
The spectral gap of the Harmonic oscillator

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leads to

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\int_{\mathbb{R}} x^2 |f(x)|^2 \, dx + \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \geq \frac{1}{2} \int_{\mathbb{R}} |f(x)|^2 \, dx,
\]

or, which is equivalent,

\[
\left\langle x^2 f - \left( \frac{d}{dx} \right)^2 f, f \right\rangle \geq \frac{1}{2} \| f \|_2^2.
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\(1/2\) is the smallest eigenvalue of the harmonic oscillator

\(H := x^2 - \left( \frac{d}{dx} \right)^2\) and \(e^{-x^2/2}\) is the corresponding eigenfunction.
The eigenvalues of the Harmonic oscillator

\[ n + \frac{1}{2} \] is the \( n \)th eigenvalue of the harmonic oscillator
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Moreover, \( n + \frac{1}{2} \) is equal to

\[
\max \min \{ \langle Hf, f \rangle ; \ f \in (f_1, \cdots, f_n)^\perp, \|f\|_2 = 1 \}.
\]

Extrema are given by Hermite functions.
Slepian Extremal problem

How large on $[-1, +1]$ are functions with spectrum in $[-c, +c]$?
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Then

$$\lambda_0(c) = \max \left\{ \int_{-1}^{+1} |f|^2 dx ; \quad \|f\|_2 = 1, \text{Supp}(\hat{f}) \subset [-c, +c] \right\}.$$
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Furthermore

$$\lambda_n(c) = \min_{f_1, \cdots, f_n} \max \left\{ \int_{-1}^{-1} |f|^2 \, dx \mid f \in (f_1, \cdots, f_n)^\perp, \|f\|_2 = 1 \right\}. $$
Slepian Extremal problem

How large on \([-1, +1]\) are functions with spectrum in \([-c, +c]\)?

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\]

Consider the operator on \(L^2([-1, +1])\)

\[
\mathcal{F}_c(f)(x) := \left( \frac{c}{2\pi} \right)^{1/2} \int_{-1}^{+1} e^{ix\xi} f(\xi) \, d\xi.
\]

Then \(\lambda_n(c)\) are the eigenvalues of \(\mathcal{F}_c^* \mathcal{F}_c\).

Key point: \(\mathcal{F}_c^* \mathcal{F}_c(f) = f \ast \chi_{[-c,+c]}\) is compact.
The asymptotic behavior of $\lambda_n(c)$. 

(Landau and Widom 1980) 
Asymptotically for $c$ tending to $\infty$, the sequence $\lambda_n(c)$ stays close to 1 for $n \leq \frac{2}{\pi} c$, then decreases exponentially rapidly.
The asymptotic behavior of \( \lambda_n(c) \).

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Asymptotically for \( c \) tending to \( \infty \), the sequence \( \lambda_n(c) \) stays close to 1 for \( n \leq \frac{2}{\pi} c \), then decreases exponentially rapidly.

Let \( \psi_{n,c} \) be the eigenfunction corresponding to \( \lambda_{n,c} \).

The main tool: (Slepian) The eigenfunctions \( \lambda_n \) are also eigenfunctions of an explicit Sturm-Liouville operator on \((-1, +1)\)

\[
\mathcal{L}_c \phi := -\frac{d}{dx} \left[(1 - x^2)\phi'\right] + c^2 x^2 \phi.
\]
Why Prolate Spheroidal Wave Functions?

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Arises in finding particular “radial” solutions of the Helmotz equation \( \Delta \Phi + k^2 \Phi = 0 \) in three dimensions in the “prolate spheroidal coordinates”. Explains the name: Prolate Spheroidal Wave Functions PSWF.
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C. Niven: **On the Conduction of Heat in Ellipsoids of Revolution**, *Philosophical Trans. R. Soc. Lond.* 1880 171, 117-151. In this remarkable piece of work, Niven has developed a detailed computational and asymptotic methods for the PSWFs and the eigenvalues \( \chi_n(c) \).
What is known on PSWF?

- The $\psi_{n,c}$ are the bounded eigenfunctions on $(-1, +1)$ of

$$\mathcal{L}_c \phi := -\frac{d}{dx} [(1 - x^2)\phi'] + c^2 x^2 \phi$$

related to the eigenvalues $\chi_n(c)$.

- $\psi_{n,c}$ is of the same parity as $n$.

- They are also eigenfunctions of $\mathcal{F}_c$ : for all $x$,

$$\psi_{n,c}(x) := \mu_n \int_{-1}^{+1} e^{icx\xi} f(\xi) d\xi.$$

- They are entire functions.
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They are used for numerics (for Helmotz Equation, in Signal Processing, as an orthonormal basis for the representation of functions on $(-1, +1)$). Work of Rokhlin Xiao et al., Chen Gottlieb and Hesthaven, Wang, Boyd, Karoui, Moumni, etc.
The WKB method for $\mathcal{L}_c$.

$\psi_{n,c}$, related to the eigenvalue $\chi_n := \chi_n(c)$, is the bounded solution of

$$\frac{d}{dx} [(1 - x^2)\psi'(x)] + \chi_n (1 - qx^2) \psi(x) = 0, \quad x \in (-1, 1). \quad (1)$$

Here we assume that $q = c^2/\chi_n < 1$.
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Remark. $\chi_n$ is not known. Heuristically (and asymptotically) equivalent to $c < \frac{\pi}{2} n$. 

S$\left( x \right):= S_{q(x)} = \int_1^x \sqrt{1 - qt^2} \, dt$,
The WKB method for $\mathcal{L}_c$. 

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**Remark.** $\chi_n$ is not known. Heuristically (and asymptotically) equivalent to $c < \frac{\pi}{2} n$.

We look for $\psi$ under the form

$$
\psi(x) = (1 - x^2)^{-1/4}(1 - qx^2)^{-1/4} U(S(x)). \quad (3)
$$

$$
S(x) := S_q(x) = \int_x^1 \sqrt{\frac{1 - qt^2}{1 - t^2}} \, dt, \quad x \in [0, 1). \quad (2)
$$
The WKB method for $\mathcal{L}_c$

**Lemma.** For $q < 1$, there exists a function $F := F_q$ that is continuous on $[0, S(0)]$, satisfies the inequality

$$|F(s)| \leq \frac{3}{(1 - q)^3}, \quad (4)$$

and such that $U$ is a solution of the equation

$$U''(s) + \left( \chi_n + \frac{1}{4s^2} \right) U(s) = F(s)U(s), \quad s \in [0, S(0)]. \quad (5)$$
The WKB method for $\mathcal{L}_C$.

The homogeneous equation

$$U''''(s) + \left( \chi_n + \frac{1}{4s^2} \right) U(s) = 0$$

has the two independent solutions

$$U_1(s) = \chi_n^{1/4} \sqrt{s} J_0(\sqrt{\chi_n s}), \quad U_2(s) = \chi_n^{1/4} \sqrt{s} Y_0(\sqrt{\chi_n s}).$$
The WKB method for $\mathcal{L}_C$.

The homogeneous equation

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The function $U$ is given by

$$U = AU_1 + R$$

with $\sqrt{\chi_n} R$ a bounded function.
**Theorem.** [A.B., A.K. (2010)] For \( q = c^2 / \chi_n(c) < 1 \) and \( 0 \leq x \leq 1 \),

\[
\psi_{n,c}(x) = A \frac{\chi_n(c)^{1/4} \sqrt{S_q(x) J_0(\sqrt{\chi_n(c)} S_q(x))}}{(1 - x^2)^{1/4}(1 - qx^2)^{1/4}} + R_{n,c}(x) \tag{6}
\]

with

\[
\sup_{x \in [0,1]} |R_{n,c}(x)| \leq C_q \chi_n(c)^{-1/2}. \tag{7}
\]
Uniform approximation of PSWF

**Theorem.** [A.B., A.K. (2010)] For $q = c^2/\chi_n(c) < 1$ and $0 \leq x \leq 1$,

$$\psi_{n,c}(x) = A\frac{\chi_n(c)^{1/4} \sqrt{S_q(x)} J_0(\sqrt{\chi_n(c)} S_q(x))}{(1 - x^2)^{1/4}(1 - qx^2)^{1/4}} + R_{n,c}(x) \quad (6)$$

with

$$\sup_{x \in [0,1]} |R_{n,c}(x)| \leq C_q \chi_n(c)^{-1/2}. \quad (7)$$

Close approximations were known, but for fixed $x$. 
Uniform bounds of the PSWFs

\[ \sup_{|x| \leq 1} |\psi_n(x)| \leq C_q \chi_n^{1/4}, \]

with the maximum obtained at 1.

\[ \sup_{|x| \leq 1} (1 - x^2)^{1/4} |\psi_n(x)| \leq C_q. \]
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\]

\[
\sup_{x \in [-1,1]} |\psi'_n(x)| \leq C \chi_n^{5/4},
\]

\[
\sup_{x \in [-1,1]} (1 - x^2) |\psi'_n(x)| \leq C \sqrt{\chi_n}.
\]

The method does not give the derivatives of higher order.
The main difficulty: estimate the constant $A$, using the fact that $\psi_n$ has norm 1.
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For the derivative, use of the equation:

$$(1 - x^2)\psi'_n(x) = \chi_n \int_x^1 (1 - qt^2)\psi_n(t) \, dt.$$
The exponential decay of $\lambda_n(c)$.

Well-known equality [Slepian 1964, Rokhlin et al. (2007)] : that for any positive integer $n$, we have $\lambda_n(c) = \lambda' \times \lambda''$, with

$$\lambda' : \quad = \quad \frac{c^{2n+1}(n!)^4}{2((2n)!)^2(\Gamma(n+3/2))^2}$$

$$\lambda'' : \quad = \quad \exp \left( 2 \int_0^c \frac{(\psi_{n,\tau}(1))^2 - (n + 1/2)}{\tau} \, d\tau \right).$$

(8) (9)

Numerical evidence, see [Rokhlin et al. 2007], indicates that $(\psi_{n,\tau}(1))^2 - (n + 1/2) \leq 0, \forall \, t \geq 0$. If we accept this assertion, then we observe that the sequence $\lambda_n(c)$ decays faster than $c \left( \frac{ec}{4n} \right)^{2n}$ so that the exponential decay has started at $[ec/4]$. 
The exponential decay of $\lambda_n(c)$. Well-known equality [Slepian 1964, Rokhlin et al. (2007)]: that for any positive integer $n$, we have $\lambda_n(c) = \lambda' \times \lambda''$, with

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To compare to [Landau and Widom 1980]: $\forall c > 0, \forall 0 < \alpha < 1$, $N(\alpha) = \#\{\lambda_i(c); \lambda_i(c) > \alpha\}$ is given by

$$N(\alpha) = \frac{2c}{\pi} + \left[\frac{1}{\pi^2} \log \left(\frac{1 - \alpha}{\alpha}\right)\right] \log(c) + o(\log(c)).$$
The exponential decay of $\lambda_n(c)$. 

(A.B., A.K. (2010)) : Let $\delta > 0$. There exists $N$ and $\kappa$ such that, for all $c \geq 0$ and $n \geq \max(N, \kappa c)$,

$$\lambda_n(c) \leq e^{-\delta(n-\kappa c)}.$$
The exponential decay of $\lambda_n(c)$.

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Work in progress :

- Bounds below for the $\lambda_n(c)$.
- Sharp estimates for the $\chi_n(c)$.
- Qualitative properties of the PSWFs.
Almost band-limited functions.

Let $T$ and $\Omega$ de two measurable sets. A function pair $(f, \hat{f})$ is said to be $\epsilon_T$–concentrated in $T$ and $\epsilon_\Omega$–concentrated in $\Omega$ if

$$\int_{T^c} |f(t)|^2 \, dt \leq \epsilon_T^2, \quad \int_{\Omega^c} |\hat{f}(\omega)|^2 \, d\omega \leq \epsilon_\Omega^2.$$
**Approximation of almost band-limited functions.**

*Theorem*

If $f$ is an $L^2$ normalized function that is $\epsilon_T-$concentrated in $T = [-1, +1]$ and $\epsilon_\Omega-$band concentrated in $\Omega = [-c, +c]$, then for any positive integer $N$, we have

$$\left( \int_{-1}^{+1} |f - S_N f|^2 dt \right)^{1/2} \leq \epsilon_\Omega + \sqrt{\lambda_N(c)}$$

(10)

and, as a consequence,

$$\|f - \chi_{[-T,+T]} S_N f\|_2 \leq \epsilon_T + \epsilon_\Omega + \sqrt{\lambda_N(c)}.$$  

(11)
Key point: for $f$ whose Fourier transform is supported on $[-c, +c]$,

$$f = \sum_{n} a_n \psi_n, \quad f - S_N f = \sum_{n \geq N} a_n \psi_n.$$
Key point: for $f$ whose Fourier transform is supported on $[-c, +c]$, 

$$f = \sum_n a_n \psi_n, \quad f - S_N f = \sum_{n \geq N} a_n \psi_n.$$ 

So 

$$\int_{-1}^{+1} |f - S_N f|^2 dt = \sum_{n \geq N} |a_n|^2 \leq \lambda_N(c) \sum_{n \geq N} \frac{|a_n|^2}{\lambda_n(c)} \leq \lambda_N(c) \|f\|_2^2.$$
**Theorem**

Let be a positive real number. Assume that $f \in H^s([-1, +1])$, for some positive real number $s > 0$. Then for any $c \geq 0$ and any integer $N \geq 1$, we have

\[
\|f - S_Nf\|_2 \leq K(1 + c^2)^{-s/2}\|f\|_{H^s} + K\sqrt{\lambda_N(c)}\|f\|_2. \tag{12}
\]

Here, the constant $K$ depends only on $s$. Moreover it can be taken equal to 1 when $f$ belongs to the space $H^s_0([-1, +1])$. 