Uncertainty Principle and Prolate Spheroidal Wave functions

Aline Bonami,
Université d'Orléans

Joint work with Abderrazek Karoui

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Heisenberg Uncertainty Principle

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Mathematically, this is equivalent to the inequality

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 \, dx\right)^{1/2} \times \left(\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2} \geq \int_{\mathbb{R}} |f(x)|^2 \, dx,$$

with equality for Gaussian functions.

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} f(x) dx.$$

The spectral gap of the Harmonic oscillator

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leads to

$$\int_{\mathbb{R}} x^2 |f(x)|^2\,dx + \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2\,d\xi \geq \frac{1}{2}\int_{\mathbb{R}} |f(x)|^2\,dx,$$

or, which is equivalent,

$$\left\langle x^2 f - \left(\frac{d}{dx}\right)^2 f, f \right\rangle \geq \frac{1}{2}\|f\|_2^2.$$

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$1/2$ is the smallest eigenvalue of the harmonic oscillator $H := x^2 - \left(\frac{d}{dx}\right)^2$ and $e^{-x^2/2}$ is the corresponding eigenfunction.

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$n + \frac{1}{2}$ is the nth eigenvalue of the harmonic oscillator $H := x^2 - \left(\frac{d}{dx}\right)^2$ and $H(x)e^{-x^2/2}$ is the corresponding eigenfunction.

Moreover, $n + \frac{1}{2}$ is equal to

$$\max_{f_1,\cdots f_n} \min\{\langle Hf, f\rangle \; ; \; f \in (f_1, \cdots, f_n)^{\perp}, \|f\|_2 = 1\}.$$

Extrema are given by Hermite functions.

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How large on $[-1,+1]$ are functions with spectrum in $[-c,+c]$?

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Consider the operator on $L^2([-1,+1])$

$$\mathcal{F}_c(f)(x) := \left(\frac{c}{2\pi}\right)^{1/2} \int_{-1}^{+1} e^{icx\xi} f(\xi) d\xi.$$

Then $\lambda_n(c)$ are the eigenvalues of $\mathcal{F}_c^*\mathcal{F}_c$.
Key point : $\mathcal{F}_c^*\mathcal{F}_c(f) = f * \widehat{\chi_{[-c,+c]}}$ is compact.

The asymptotic behavior of $\lambda_n(c)$.

(Landau and Widom 1980)
Asymptotically for c tending to ∞, the sequence $\lambda_n(c)$ stays close to 1 for $n \leq \frac{2}{\pi} c$, then decreases exponentially rapidly.

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Let $\psi_{n,c}$ be the eigenfunction corresponding to $\lambda_{n,c}$.

The main tool : (Slepian) The eigenfunctions λ_n are also eigenfunctions of an explicit Sturm-Liouville operator on $(-1, +1)$

$$\mathcal{L}_c \phi := -\frac{d}{dx}\left[(1-x^2)\phi'\right] + c^2 x^2 \phi.$$

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C. Niven : **On the Conduction of Heat in Ellipsoids of Revolution,** *Philosophical Trans. R. Soc. Lond. 1880 171, 117-151.* In this remarkable piece of work, Niven has developed a detailed computational and asymptotic methods for the PSWFs and the eigenvalues $\chi_n(c)$.

What is known on PSWF ?

- The $\psi_{n,c}$ are the bounded eigenfunctions on $(-1,+1)$ of

$$\mathcal{L}_c\phi := -\frac{d}{dx}\left[(1-x^2)\phi'\right] + c^2x^2\phi$$

  related to the eigenvalues $\chi_n(c)$.
- $\psi_{n,c}$ is of the same parity as n.
- They are also eigenfunctions of $\mathcal{F}_c$: for all x,

$$\psi_{n,c}(x) := \mu_n \int_{-1}^{+1} e^{icx\xi} f(\xi) d\xi.$$

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They are used for numerics (for Helmotz Equation, in Signal Processing, as an orthonormal basis for the representation of functions on $(-1,+1)$). Work of Rokhlin Xiao et al., Chen Gottlieb and Hesthaven, Wang, Boyd, Karoui, Moumni, etc.

The WKB method for $\mathcal{L}_c$.

$\psi_{n,c}$, related to the eigenvalue $\chi_n := \chi_n(c)$, is the bounded solution of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\psi'(x)\right] + \chi_n(1-qx^2)\psi(x) = 0, \quad x \in (-1,1). \quad (1)$$

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$$S(x) := S_q(x) = \int_x^1 \sqrt{\frac{1-qt^2}{1-t^2}}\, dt, \quad x \in [0,1). \quad (2)$$

We look for ψ under the form

$$\psi(x) = (1-x^2)^{-1/4}(1-qx^2)^{-1/4}U(S(x)). \quad (3)$$

The WKB method for $\mathcal{L}_c$

Lemma. For $q < 1$, there exists a function $F := F_q$ that is continuous on $[0, S(0)]$, satisfies the inequality

$$|F(s)| \leq \frac{3}{(1-q)^3}, \tag{4}$$

and such that U is a solution of the equation

$$U''(s) + \left(\chi_n + \frac{1}{4s^2}\right) U(s) = F(s)U(s), \quad s \in [0, S(0)]. \tag{5}$$

The WKB method for $\mathcal{L}_c$.

The homogeneous equation

$$U''(s) + \left(\chi_n + \frac{1}{4s^2}\right) U(s) = 0$$

has the two independent solutions

$$U_1(s) = \chi_n^{1/4} \sqrt{s} J_0(\sqrt{\chi_n} s), \quad U_2(s) = \chi_n^{1/4} \sqrt{s} Y_0(\sqrt{\chi_n} s).$$

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The function U is given by

$$U = AU_1 + R$$

with $\sqrt{\chi_n}R$ a bounded function.

Uniform approximation of PSWF

Theorem.[A.B., A.K. (2010)] For $q = c^2/\chi_n(c) < 1$ and $0 \leq x \leq 1$,

$$\psi_{n,c}(x) = A\frac{\chi_n(c)^{1/4}\sqrt{S_q(x)}J_0(\sqrt{\chi_n(c)}S_q(x))}{(1-x^2)^{1/4}(1-qx^2)^{1/4}} + R_{n,c}(x) \qquad (6)$$

with

$$\sup_{x\in[0,1]} |R_{n,c}(x)| \leq C_q \chi_n(c)^{-1/2}. \qquad (7)$$

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Close approximations were known, but for fixed x.

Uniform bounds of the PSWFs

$$\sup_{|x|\leq 1} |\psi_n(x)| \leq C_q \chi_n^{1/4},$$

with the maximum obtained at 1.

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$$\sup_{x\in[-1,1]} |\psi_n'(x)| \leq C {\chi_n}^{5/4},$$

$$\sup_{x\in[-1,1]} (1-x^2)|\psi_n'(x)| \leq C\sqrt{\chi_n}.$$

The method does not give the derivatives of higher order.

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For the derivative, use of the equation :

$$(1-x^2)\psi_n'(x) = \chi_n \int_x^1 (1-qt^2)\psi_n(t)\,dt.$$

The exponential decay of $\lambda_n(c)$.

Well-known equality [Slepian 1964, Rokhlin et al. (2007)] : that for any positive integer n, we have $\lambda_n(c) = \lambda' \times \lambda''$, with

$$\lambda' : = \frac{c^{2n+1}(n!)^4}{2((2n)!)^2(\Gamma(n+3/2))^2} \tag{8}$$

$$\lambda'' : = \exp\left(2\int_0^c \frac{(\psi_{n,\tau}(1))^2 - (n+1/2)}{\tau}\, d\tau\right). \tag{9}$$

Numerical evidence, see [Rokhlin et al. 2007], indicates that $(\psi_{n,\tau}(1))^2 - (n+1/2) \leq 0, \forall\, t \geq 0$. If we accept this assertion, then we observe that the sequence $\lambda_n(c)$ decays faster than $\frac{c}{2}\left(\frac{ec}{4n}\right)^{2n}$ so that the exponential decay has started at $[ec/4]$.

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To compare to [Landau and Widom 1980] : $\forall c > 0$, $\forall\, 0 < \alpha < 1$, $N(\alpha) = \#\{\lambda_i(c);\ \lambda_i(c) > \alpha\}$ is given by

$$N(\alpha) = \frac{2c}{\pi} + \left[\frac{1}{\pi^2}\log\left(\frac{1-\alpha}{\alpha}\right)\right]\log(c) + o(\log(c)).$$

The exponential decay of $\lambda_n(c)$.

(A.B., A.K. (2010)) : Let $\delta > 0$. There exists N and κ such that, for all $c \geq 0$ and $n \geq \max(N, \kappa c)$,

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Work in progress :

- Bounds below for the $\lambda_n(c)$.
- Sharp estimates for the $\chi_n(c)$.
- Qualitative properties of the PSWFs.

Almost band-limited functions.

Let T and Ω de two measurable sets. A function pair $(f, \widehat{f})$ is said to be ϵ_T-concentrated in T and $\epsilon_\Omega-$concentrated in Ω if

$$\int_{T^c} |f(t)|^2 \, dt \leq \epsilon_T^2, \qquad \int_{\Omega^c} |\widehat{f}(\omega)|^2 \, d\omega \leq \epsilon_\Omega^2.$$

Approximation of almost band-limited functions.

Theorem

If f is an L^2 normalized function that is ϵ_T-concentrated in $T = [-1, +1]$ and $\epsilon_\Omega-$band concentrated in $\Omega = [-c, +c]$, then for any positive integer N, we have

$$\left(\int_{-1}^{+1} |f - S_N f|^2 dt \right)^{1/2} \leq \epsilon_\Omega + \sqrt{\lambda_N(c)} \qquad (10)$$

and, as a consequence,

$$\| f - \chi_{[-T,+T]} S_N f \|_2 \leq \epsilon_T + \epsilon_\Omega + \sqrt{\lambda_N(c)}. \qquad (11)$$

Key point : for f whose Fourier transform is supported on $[-c, +c]$,

$$f = \sum_n a_n \psi_n, \qquad\qquad f - S_N f = \sum_{n \geq N} a_n \psi_n.$$

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So

$$\begin{aligned} \int_{-1}^{+1} |f - S_N f|^2 dt &= \sum_{n \geq N} |a_n|^2 \\ &\leq \lambda_N(c) \sum_{n \geq N} \frac{|a_n|^2}{\lambda_n(c)} \\ &\leq \lambda_N(c) \|f\|_2^2. \end{aligned}$$

Approximation in Sobolev spaces.

Theorem

Let be a positive real number. Assume that $f \in H^s([-1,+1])$, for some positive real number $s > 0$. Then for any $c \geq 0$ and any integer $N \geq 1$, we have

$$\|f - S_N f\|_2 \leq K(1+c^2)^{-s/2}\|f\|_{H^s} + K\sqrt{\lambda_N(c)}\|f\|_2. \tag{12}$$

Here, the constant K depends only on s. Moreover it can be taken equal to 1 when f belongs to the space $H_0^s([-1,+1])$.