Energy critical heat equation in two space dimensions

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joint work with
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Outline

Introduction

Consider the IVP

$$\begin{cases} u_t - \Delta u = f(u) \\ u(0) = u_0 \end{cases} \tag{1}$$

- $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$.
- $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ with $f(0) = 0$.
- $u_0 \in L^\infty \implies \exists\ T(u_0) > 0$ and a unique solution $u \in \mathcal{C}([0,T[; L^\infty)$ to (1).
- The Cauchy problem (1) has been extensively studied in the scale of Lebesgue spaces L^q, especially for polynomial type nonlinearities i.e.

$$f(u) := \pm|u|^{\gamma-1}u, \quad \gamma > 1. \tag{2}$$

- Scaling invariance:

$$u(t,x) \Longrightarrow u_\lambda(t,x) := \lambda^{2/\gamma}\, u(\lambda^2 t, \lambda x), \quad \lambda > 0\ .$$

- The Lebesgue space $L^{q_c}(\mathbb{R}^d)$ with index $q_c := \frac{d(\gamma-1)}{2}$ is the only one invariant under the same scaling.
- **Subcritical case**: $q \geq q_c \geq 1$
  - Weissler [1980]: Existence and uniqueness in $\mathcal{C}_T(L^q) \cap L^\infty_{loc}(L^\infty)$.
  - Brezis-Cazenave [1996]: Unconditional uniqueness.
- **Critical case**: $q = q_c$ and $d \geq 3$. There are two sub-cases:
  - $q_c > \gamma + 1$, or equivalently $\gamma > \frac{d+2}{d-2}$. The existence was proved by Weissler and the uniqueness by Brezis-Cazenave.
  - $q = q_c = \gamma + 1$, or equivalently $q = \frac{2d}{d-2}$ and $\gamma = \frac{d+2}{d-2}$ (double critical case).
    - Weissler [1981]: Conditional wellposedness.
    - Ni-Sacks [1985]: Nonuniqueness where the underlying space is the unit ball.
    - Terraneo [2002]: Nonuniqueness for the whole space and for suitable data.
    - Matos-Terraneo [2003]: Nonuniqueness for general data.

- **Supercritical case**: $q < q_c$.

  • Haraux-Weissler [1982]: Uniqueness is lost for initial data $u_0 = 0$ and for $1 + \frac{1}{d} < \gamma < \frac{d+2}{d-2}$.

  • There exists no (local) solution in any reasonable weak sense.

- Ribaud [1998]: Wellposedness in Sobolev spaces H^s_p.

- Miao-Zhang [2004]: Wellposedness in Besov spaces.

Two dimensional case

- One of the results of Miao-Zhang is the global well posedness in $H^1(\mathbb{R}^2)$ for small data, and for nonlinearities f satisfying

$$|f'(u)| \leq C|u|^2 \, \mathrm{e}^u \, .$$

- Recall that the Sobolev space $H^1(\mathbb{R}^2)$ is embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for every $2 \leq p < \infty$ but not in $L^\infty(\mathbb{R}^2)$.
- The optimal (critical) Sobolev embedding is known to be

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2), \tag{3}$$

where $\mathcal{L}(\mathbb{R}^2)$ is the Orlicz space associated to the function $\phi(s) = \mathrm{e}^{s^2} - 1$

- A natural question to ask then is: what is the critical nonlinearity in dimension $d = 2$?

A natural model to investigate in 2D is

$$\begin{cases} u_t - \Delta u = \pm u(\mathrm{e}^{u^2} - 1) \quad \text{in} \quad \mathbb{R}^2 \\ u(0) = u_0 \end{cases} \tag{4}$$

- **GoaL**: Existence and/or uniqueness of local/global solutions to (4) when the data is no longer in L^∞.

- The largest Lebesgue type space in which the equation is meaningful in the distributional sense is of Orlicz kind.

- Ruf & Terraneo [2002]: Local existence for small data in Orlicz space.

Main results

Good H^1- Theory

Theorem

Let $u_0 \in H^1(\mathbb{R}^2)$.

1. *There exists a unique solution u to (4) in $\mathcal{C}([0,T], H^1)$.*
2. *If $f(u) = -u(\mathrm{e}^{u^2} - 1)$, then the (above) solution is global.*
3. *If $f(u) = u(\mathrm{e}^{u^2} - 1)$, $u_0 \neq 0$ and $J(u_0) \leq 0$, then the (above) solution blows up in finite time.*

Here the energy J is given by

$$J(u(t)) \stackrel{\text{def}}{=} \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)} - \int_{\mathbb{R}^2} F(u(t))\, dx,$$

where

$$F(u) = \int_0^u f(v)\, dv\,.$$

Nonuniqueness

Theorem

Let B_1 be the unit ball of $\mathbb{R}^2$. There exists infinitely many $u_0 \in \mathcal{L}(B_1)$ such that the Cauchy problem

$$\begin{cases} u_t - \Delta u = u(\mathrm{e}^{u^2} - 1) \quad \text{in} \quad B_1 \\ u(0) = u_0 \quad \text{in} \quad B_1 \\ u_{|\partial B_1} = 0 \quad \text{for} \quad t > 0 \end{cases} \tag{5}$$

has at least two solutions.

Background material

- $\|\mathrm{e}^{t\Delta}\varphi\|_{L^\gamma} \leq C\, t^{\frac{1}{\gamma}-\frac{1}{\beta}}\, \|\varphi\|_{L^\beta},\; t>0,\; 1\leq\beta\leq\gamma\leq\infty\,.$

- Using Duhamel's formula for the equation $u_t - \Delta u = F(t,x)$, we deduce

$$\begin{aligned}
\|u\|_{L^\infty_T(H^1)} &\leq C\Big(\|u(0)\|_{H^1} + \|F\|_{L^1_T(H^1)}\Big) \\
\|u\|_{L^\infty_T(H^1)} &\leq C\Big(\|u(0)\|_{H^1} + \|F\|_{L^1_T(L^2)} + T^{1/2}\,\|\nabla F\|_{L^\infty_T(L^1)}\Big) \\
\|u\|_{L^\infty_T(L^\infty)} &\leq C\Big(\|u(0)\|_{L^\infty} + \|F\|_{L^1_T(L^\infty)}\Big)
\end{aligned}$$

In order to control the nonlinear part in $L^1_t(H^1_x)$, we will use the following Moser-Trudinger type inequalities.

- $\displaystyle\int_{\mathbb{R}^2} \left(\mathrm{e}^{\alpha|u(x)|^2} - 1\right) dx \leq C_\alpha \|u\|^2_{L^2(\mathbb{R}^2)}$ whenever $0 \leq \alpha < 4\pi$ and $u \in H^1$ satisfying $\|\nabla u\|_{L^2} \leq 1$. The above inequality is false for $\alpha \geq 4\pi$.

- If we require $\|u\|_{H^1} \leq 1$ rather than $\|\nabla u\|_{L^2} \leq 1$, we obtain

$$\sup_{\|u\|_{H^1} \leq 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2} - 1\right) dx < \infty\,.$$

The above property is false for $\alpha > 4\pi$.

Definition

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function such that

$$\phi(0) = 0 = \lim_{s \to 0^+} \phi(s), \quad \lim_{s \to \infty} \phi(s) = \infty.$$

The Orlicz space L^ϕ is defined via the Luxembourg norm

$$\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$

- $\phi(s) = s^p$, $1 \leq p < \infty \Longrightarrow L^\phi = L^p$.
- $\phi_\alpha(s) = \mathrm{e}^{\alpha s^2} - 1 \Longrightarrow L^{\phi_\alpha} = L^{\phi_1} = \mathcal{L}$.

We shall use the following properties of Orlicz spaces.

- If $T : L^1 \to L^1$ with norm M_1 and $T : L^\infty \to L^\infty$ with norm M_∞, then $T : L^\phi \to L^\phi$ with norm

$$M \leq \kappa_{\phi,d} \sup(M_1, M_\infty),$$

where $\kappa_{\phi,d}$ depends only on ϕ and the dimension d.

- For any $p \geq 2$, $\mathcal{L}(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ and we have

$$\|u\|_{L^p} \leq \Big(\Gamma(\frac{p}{2}+1)\Big)^{\frac{1}{p}} \|u\|_{\mathcal{L}}\,.$$

Recall the following parabolic regularizing effect due to Brezis-Cazenave that we will use to obtain a locally (in time) bounded solution to (4) with singular data. Consider the following linear heat equation with potential

$$\begin{cases} u_t - \Delta u - a(t,x)u = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \\ u(0) = u_0, \end{cases} \tag{6}$$

where Ω is a smooth bounded domain of $\mathbb{R}^2$.

Theorem (Brezis-Cazenave)

Let $0 < T < \infty$, $\sigma > 1$, and let $a \in L^\infty([0,T]; L^\sigma)$. Given $u_0 \in L^r$, $1 \leq r < \infty$, there exists a unique solution $u \in \mathcal{C}([0,T]; L^r) \cap L^\infty_{loc}([0,T]; L^\infty)$ of equation (6).

Sketches of the proofs

Local existence in H^1

- We decompose the initial data $u_0 \in H^1$ as

$$u_0 = u_{0,N} + u_0^N \quad (N \text{ large})$$

$$= H^1 \cap L^\infty + \text{small in } H^1$$

- We solve the IVP with data $u_{0,N}$ to obtain a local solution v.

- To recover a solution of our original problem we solve a perturbed equation satisfied by $w := u - v$ with small data u_0^N.

Unconditional uniqueness

Let $u,\ v \in \mathcal{C}([0,\ T]; H^1)$ be two solutions to (4) with same data u_0 and set $w := u - v$. Then

$$w(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} a(s) w(s)\, ds,$$

where the potential a is given by

$$a(t,x) := \begin{cases} \frac{f(u)-f(v)}{w} & \text{if} \quad w \neq 0 \\ \\ f'(u) & \text{if} \quad w = 0 \end{cases}$$

Remark that $w \in L^\infty(0,\ T; L^q)$ for any $2 \leq q < \infty$.

To prove that $w \equiv 0$ on $[0, T]$, we use the following lemma which can be seen as an extension to dimension two of Brezis-Cazenave's results about dimension $d \geq 3$.

Lemma

Let $a \in \mathcal{C}([0, T]; L^p(\mathbb{R}^2))$ *and* $u \in L^\infty((0, T); L^q(\mathbb{R}^2))$ *with* $2 \leq q < \infty, 1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} \neq 1$, *and such that*

$$u(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} a(s) u(s) ds, \quad 0 \leq t \leq T.$$

Then $u = 0$ *on* $[0, T]$.

A crucial fact here is that the potential is continuous in time. We don't know how to extend this lemma to the case when the potential is only L^∞ *in time.*

- The uniqueness follows once we prove that $a \in \mathcal{C}([0, T]; L^p(\mathbb{R}^2))$ for some $1 < p < \infty$.

- This basically follows from the fact that

$$u \in \mathcal{C}([0, T]; H^1(\mathbb{R}^2)) \implies \mathrm{e}^{u^2} - 1 \in \mathcal{C}([0, T]; L^1(\mathbb{R}^2)) .$$

Global existence

Here we assume that $f(u) = -u\left(e^{u^2} - 1\right)$.

Proposition

Assume that $u_0 \in H^1(\mathbb{R}^2)$ *and* $u \in \mathcal{C}([0, T); H^1(\mathbb{R}^2))$ *solution to* (4). *Then*

$$\|u(t)\|_{L^\infty} \leq \sqrt{2}\,\|u_0\|_{L^2}, \qquad 0 < t < T.$$

We use the level set energy inequality for $c_k := M(1 - 2^{-k})$, where $M > 0$ will be chosen later. This idea was used by Caffarelli-Vasseur (**Annals of Mathematics (2010)**) for the quasi-geostrophic equation.

- This leads to the following energy inequality for the level set function $u_k \stackrel{\text{def}}{=} (u - c_k)_+$:

$$\frac{d}{dt}\int_{\mathbb{R}^2} |u_k(t)|^2\, dx + 2\int_{\mathbb{R}^2} |\nabla u_k(t)|^2 dx \leq 0\,. \qquad (7)$$

- Let $t_0 > 0$, and denote by $T_k \stackrel{\text{def}}{=} t_0(1 - 2^{-k})$ and

$$U_k \stackrel{\text{def}}{=} \sup_{t \geq T_k} \Big(\int_{\mathbb{R}^2} |u_k(t,x)|^2\, dx\Big) + 2\int_{T_k}^{+\infty}\int_{\mathbb{R}^2} |\nabla u_k(t,x)|^2 dxdt\,.$$

- By integrating (7) in time, and using some interpolation estimates, we obtain finally

$$U_k \;\leq\; A\, C^{k-1}\, U_{k-1}^{\frac{\alpha}{2}},$$

where

$$A \stackrel{\text{def}}{=} 2^{\alpha/2+4}\, M^{-\alpha}\, t_0^{-1}\, \|u_0\|_{L^2}^2, \quad \text{and} \quad C \stackrel{\text{def}}{=} 2^{\alpha+1}\,.$$

- Taking $\alpha > 2$ and $M \stackrel{\text{def}}{=} 2^{\frac{\alpha^2+10\alpha-12}{2\alpha(\alpha-2)}}\, t_0^{-\frac{1}{\alpha}}\, \|u_0\|_{L^2}$, we get

$$\lim_{k\to\infty} U_k = 0\,.$$

- The conclusion follows by letting $\alpha \to \infty$.

Blowing-up solutions

Claim: *If* $u_0 \in H^1 \backslash \{0\}$ *with* $J(u_0) \leq 0$, *then* $T^* < \infty$.

The proof is quite standard. Set $y(t) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^t \|u(s)\|_{L^2}^2 \, ds$.

Since $\Big(uf(u) - 2F(u)\Big) \geq \varepsilon F(u)$ for some $\varepsilon > 0$, we find that

$$y(t)y''(t) \geq \ (1+\eta)\,(y'(t))^2 \quad \text{for some} \quad \eta = \eta(\varepsilon) > 0\,.$$

The fact that this ordinary differential inequality blows up in finite time ensures the claim.

Nonuniqueness

The proof is done in four steps. Set $\Omega \stackrel{\text{def}}{=} B(0,1) \subset \mathbb{R}^2$.

Step 1: We construct a singular solution Q to

$$\begin{cases} -\Delta Q = f(Q) & \text{in} \quad \Omega, \\ Q = 0 & \text{on} \quad \partial\Omega, \\ Q(0) = \infty, & \text{and} \quad Q > 0. \end{cases}$$

We look for singular solutions which are radially symmetric. Letting $|x| \stackrel{\text{def}}{=} r = \mathrm{e}^{-t}$, $y(t) = Q(\mathrm{e}^{-t})$, we obtain

$$\begin{cases} -y''(t) = \mathrm{e}^{-2t} f(y(t)), & t \geq 0 \\ y(0) = 0, & y(\infty) = \infty. \end{cases}$$

To solve the above problem, we proceed as follows.

- For any $\alpha > 0$, we consider the Cauchy problem

$$(\mathcal{P}_\alpha) \begin{cases} -y''(t) = \mathrm{e}^{-2t} f(y(t)), & t \geq 0 \\ y(0) = 0, & y'(0) = \alpha, \end{cases}$$

  and the associated elliptic problem

$$(\mathcal{E}_\alpha) \begin{cases} -\Delta u_\alpha = f(u_\alpha), & 0 < r < 1 \\ u_\alpha(1) = 0, & u_\alpha(0) = y_\alpha(\infty). \end{cases}$$

- Elliptic regularity $+$ Existence and uniqueness of classical radial solution $\Longrightarrow \exists!\ \alpha_0$ s.t. $\lim_{t\to\infty} y_{\alpha_0}(t) = \ell \in (0, \infty)$.

Set

$$
\begin{aligned}
T(\alpha) &= \sup\Big\{s \geq 0\,; \quad y_\alpha > 0 \ \text{ on } \ (0,s)\Big\}\\
I &= \Big\{\,\alpha > 0\,; \quad T(\alpha) < \infty\,\Big\}\\
J &= \Big\{\,\alpha > 0\,; \quad \alpha \notin I\,\Big\}
\end{aligned}
$$

The end of the proof of existence consists on showing that J is a non trivial interval.

Step 2: We prove that

- $-\Delta Q = f(Q)$ in $\mathcal{D}'(\Omega)$.
- $Q \in \mathcal{L}(\Omega)$.
- $\lim_{r\to 0} \|Q\|_{\mathcal{L}(|x|<r)} = 0$.

The above classical solution in $\Omega\backslash\{0\}$ is extendable to a distribution solution in Ω thanks to the following technical lemma.

Lemma

Let $u \in C^2(\Omega\backslash\{0\})$, $u \geq 0$, *such that* $-\Delta u = f(u)$ *in* $\Omega\backslash\{0\}$. *Then*

1. $f(u) \in L^1(\Omega)$.
2. *If* $(-\log r)^\alpha u^q \in L^1(\Omega)$ *for some* $\frac{\alpha}{q-1} > 0$, *then*

$$\Delta u + f(u) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).$$

This lemma can be seen as an extension to dimension two of Ni-Sacks' result.

Step 3: We take Q as an initial data

$$\begin{cases} u_t - \Delta u = f(u) \quad \text{in} \quad \Omega \\ u(0) = Q, \;\; u(|x| = 1) = 0 \end{cases} \tag{8}$$

- We prove that (8) has a local solution $u \in L^\infty(\mathcal{L}(\Omega))$.

- Since the potential $\mathrm{e}^{u^2} - 1$ is better than L^1, we can apply Brezis-Cazenave's result about regularization effect. This leads to $u \in L^\infty(L^\infty(\Omega))$.

Step 4: Now we are able to conclude.

$$u_0 = Q \quad \longrightarrow \quad v(t,x) = Q(x)$$

$$u_0 = Q \quad \longrightarrow \quad u(t,x) \in L^\infty(L^\infty(\Omega))$$

As $Q(0) = \infty$, we find $u \neq v$.

Concluding remarks

1. We believe that there is no existence in H^s for $s < 1$. (Work in progress)

2. The blow-up analysis can be refined for positive initial energy. (Work in progress)

Thank You