Energy critical heat equation in two space dimensions

Mohamed Majdoub

joint work with
Slim Ibrahim, Rym Jrad & Tarek Saanouni

March 2011
Consider the IVP

\[
\begin{cases}
  u_t - \Delta u = f(u) \\
  u(0) = u_0
\end{cases}
\]

1. \(u(t, x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}\).
2. \(f \in C^1(\mathbb{R}, \mathbb{R})\) with \(f(0) = 0\).
3. \(u_0 \in L^\infty \implies \exists T(u_0) > 0\) and a unique solution \(u \in C([0, T[, L^\infty)\) to (1).
4. The Cauchy problem (1) has been extensively studied in the scale of Lebesgue spaces \(L^q\), especially for polynomial type nonlinearities i.e.

\[
f(u) := \pm |u|^{\gamma-1} u, \quad \gamma > 1.
\]

5. Scaling invariance:

\[
u(t, x) \implies u_\lambda(t, x) := \lambda^{2/\gamma} u(\lambda^2 t, \lambda x), \quad \lambda > 0.
\]
The Lebesgue space $L^{q_c}(\mathbb{R}^d)$ with index $q_c := \frac{d(\gamma-1)}{2}$ is the only one invariant under the same scaling.

**Subcritical case:** $q \geq q_c \geq 1$
- Weissler [1980]: Existence and uniqueness in $C_T(L^q) \cap L^\infty_{loc}(L^\infty)$.
- Brezis-Cazenave [1996]: Unconditional uniqueness.

**Critical case:** $q = q_c$ and $d \geq 3$. There are two sub-cases:
- $q_c > \gamma + 1$, or equivalently $\gamma > \frac{d+2}{d-2}$. The existence was proved by Weissler and the uniqueness by Brezis-Cazenave.
- $q = q_c = \gamma + 1$, or equivalently $q = \frac{2d}{d-2}$ and $\gamma = \frac{d+2}{d-2}$ (double critical case).
  - Ni-Sacks [1985]: Nonuniqueness where the underlying space is the unit ball.
  - Terraneo [2002]: Nonuniqueness for the whole space and for suitable data.
  - Matos-Terraneo [2003]: Nonuniqueness for general data.

Presented by: Mohamed Majdoub
• **Supercritical case**: $q < q_c$.

  • **Haraux-Weissler [1982]**: Uniqueness is lost for initial data $u_0 = 0$ and for $1 + \frac{1}{d} < \gamma < \frac{d+2}{d-2}$.

  • There exists no (local) solution in any reasonable weak sense.


• **Miao-Zhang [2004]**: Wellposedness in Besov spaces.
One of the results of Miao-Zhang is the global well posedness in $H^1(\mathbb{R}^2)$ for small data, and for nonlinearities $f$ satisfying

$$|f'(u)| \leq C|u|^2 e^u.$$ 

Recall that the Sobolev space $H^1(\mathbb{R}^2)$ is embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for every $2 \leq p < \infty$ but not in $L^\infty(\mathbb{R}^2)$.

The optimal (critical) Sobolev embedding is known to be

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2),$$

where $\mathcal{L}(\mathbb{R}^2)$ is the Orlicz space associated to the function $\phi(s) = e^{s^2} - 1$.

A natural question to ask then is: what is the critical nonlinearity in dimension $d = 2$?
A natural model to investigate in 2D is

\[
\begin{aligned}
\begin{cases}
    u_t - \Delta u = \pm u(e^{u^2} - 1) & \text{in } \mathbb{R}^2 \\
    u(0) = u_0
\end{cases}
\end{aligned}
\]  

\tag{4}

- **Goal**: Existence and/or uniqueness of local/global solutions to (4) when the data is no longer in $L^\infty$.

- The largest Lebesgue type space in which the equation is meaningful in the distributional sense is of Orlicz kind.

- **Ruf & Terraneo [2002]**: Local existence for small data in Orlicz space.
Main results
Good $H^1$- Theory

**Theorem**

Let $u_0 \in H^1(\mathbb{R}^2)$.

1. There exists a unique solution $u$ to (4) in $C([0, T], H^1)$.
2. If $f(u) = -u(e^{u^2} - 1)$, then the (above) solution is global.
3. If $f(u) = u(e^{u^2} - 1)$, $u_0 \neq 0$ and $J(u_0) \leq 0$, then the (above) solution blows up in finite time.

Here the energy $J$ is given by

$$J(u(t)) \equiv \frac{1}{2} \|\nabla u(t)\|_{L^2(\mathbb{R}^2)}^2 - \int_{\mathbb{R}^2} F(u(t)) \, dx,$$

where

$$F(u) = \int_0^u f(v) \, dv.$$
Nonuniqueness

Theorem

Let $B_1$ be the unit ball of $\mathbb{R}^2$. There exists infinitely many $u_0 \in \mathcal{L}(B_1)$ such that the Cauchy problem

$$
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= u(e^{u^2} - 1) \quad \text{in} \quad B_1 \\
    u(0) &= u_0 \quad \text{in} \quad B_1 \\
    u|_{\partial B_1} &= 0 \quad \text{for} \quad t > 0
\end{align*}
$$

has at least two solutions.
Background material
\[ \|e^{t \Delta} \varphi\|_{L^\gamma} \leq C t^{\frac{1}{\gamma} - \frac{1}{\beta}} \|\varphi\|_{L^\beta}, \quad t > 0, \ 1 \leq \beta \leq \gamma \leq \infty. \]

Using Duhamel’s formula for the equation \( u_t - \Delta u = F(t, x) \), we deduce

\[
\|u\|_{L^\infty_T(H^1)} \leq C \left( \|u(0)\|_{H^1} + \|F\|_{L^1_T(H^1)} \right) \\
\|u\|_{L^\infty_T(H^1)} \leq C \left( \|u(0)\|_{H^1} + \|F\|_{L^1_T(L^2)} + T^{1/2} \|\nabla F\|_{L^\infty_T(L^1)} \right) \\
\|u\|_{L^\infty_T(L^\infty)} \leq C \left( \|u(0)\|_{L^\infty} + \|F\|_{L^1_T(L^\infty)} \right)
\]
In order to control the nonlinear part in $L^1_t(H^1_x)$, we will use the following Moser-Trudinger type inequalities.

\[ \int_{\mathbb{R}^2} \left( e^{\alpha |u(x)|^2} - 1 \right) dx \leq C_\alpha \|u\|^2_{L^2(\mathbb{R}^2)} \]
whenever $0 \leq \alpha < 4\pi$ and $u \in H^1$ satisfying $\|\nabla u\|_{L^2} \leq 1$. The above inequality is false for $\alpha \geq 4\pi$.

If we require $\|u\|_{H^1} \leq 1$ rather than $\|\nabla u\|_{L^2} \leq 1$, we obtain

\[ \sup_{\|u\|_{H^1} \leq 1} \int_{\mathbb{R}^2} \left( e^{4\pi |u(x)|^2} - 1 \right) dx < \infty. \]

The above property is false for $\alpha > 4\pi$. 

Presented by: Mohamed Majdoub
**Definition**

Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a convex increasing function such that

$$
\phi(0) = 0 = \lim_{s \rightarrow 0^+} \phi(s), \quad \lim_{s \rightarrow \infty} \phi(s) = \infty.
$$

The Orlicz space $L^\phi$ is defined via the Luxembourg norm

$$
\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
$$

- $\phi(s) = s^p$, $1 \leq p < \infty \implies L^\phi = L^p$.
- $\phi_\alpha(s) = e^{\alpha s^2} - 1 \implies L^{\phi_\alpha} = L^{\phi_1} = L$. 

Presented by: Mohamed Majdoub

Energy critical heat equation ...
We shall use the following properties of Orlicz spaces.

- If \( T : L^1 \rightarrow L^1 \) with norm \( M_1 \) and \( T : L^\infty \rightarrow L^\infty \) with norm \( M_\infty \), then \( T : L^\phi \rightarrow L^\phi \) with norm
  \[
  M \leq \kappa_{\phi,d} \sup(M_1, M_\infty),
  \]
  where \( \kappa_{\phi,d} \) depends only on \( \phi \) and the dimension \( d \).

- For any \( p \geq 2 \), \( \mathcal{L}(\mathbb{R}^2) \subset L^p(\mathbb{R}^2) \) and we have
  \[
  \|u\|_{L^p} \leq \left( \Gamma\left(\frac{p}{2} + 1\right) \right)^{\frac{1}{p}} \|u\|_{\mathcal{L}}.
  \]
Recall the following parabolic regularizing effect due to Brezis-Cazenave that we will use to obtain a locally (in time) bounded solution to (4) with singular data. Consider the following linear heat equation with potential

\[
\begin{cases}
    u_t - \Delta u - a(t, x)u = 0 & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega, \\
    u(0) = u_0,
\end{cases}
\]

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^2$.

**Theorem (Brezis-Cazenave)**

Let $0 < T < \infty$, $\sigma > 1$, and let $a \in L^\infty([0, T]; L^\sigma)$. Given $u_0 \in L^r$, $1 \leq r < \infty$, there exists a unique solution $u \in C([0, T]; L^r) \cap L^\infty_{loc}([0, T]; L^\infty)$ of equation (6).
Sketches of the proofs
Local existence in $H^1$

- We decompose the initial data $u_0 \in H^1$ as

$$u_0 = u_{0,N} + u_0^N \quad (N \text{ large})$$

$$= H^1 \cap L^\infty + \text{small in } H^1$$

- We solve the IVP with data $u_{0,N}$ to obtain a local solution $v$.

- To recover a solution of our original problem we solve a perturbed equation satisfied by $w := u - v$ with small data $u_0^N$. 

Presented by: Mohamed Majdoub
Unconditional uniqueness

Let $u, v \in C([0, T]; H^1)$ be two solutions to (4) with same data $u_0$ and set $w := u - v$. Then

$$w(t) = \int_0^t e^{(t-s)\Delta} a(s) w(s) \, ds,$$

where the potential $a$ is given by

$$a(t, x) := \begin{cases} 
\frac{f(u) - f(v)}{w} & \text{if } w \neq 0 \\
 f'(u) & \text{if } w = 0
\end{cases}$$

Remark that $w \in L^\infty(0, T; L^q)$ for any $2 \leq q < \infty$. 

Presented by: Mohamed Majdoub
To prove that $w \equiv 0$ on $[0, T]$, we use the following lemma which can be seen as an extension to dimension two of Brezis-Cazenave’s results about dimension $d \geq 3$.

**Lemma**

Let $a \in C([0, T]; L^p(\mathbb{R}^2))$ and $u \in L^\infty((0, T); L^q(\mathbb{R}^2))$ with $2 \leq q < \infty$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} \neq 1$, and such that

$$u(t) = \int_0^t e^{(t-s)\Delta} a(s) u(s) ds, \quad 0 \leq t \leq T.$$  

Then $u = 0$ on $[0, T]$.

A crucial fact here is that the potential is continuous in time. We don’t know how to extend this lemma to the case when the potential is only $L^\infty$ in time.
The uniqueness follows once we prove that 
\( a \in C([0, T]; L^p(\mathbb{R}^2)) \) for some \( 1 < p < \infty \).

This basically follows from the fact that 
\[
    u \in C([0, T]; H^1(\mathbb{R}^2)) \implies e^{u^2} - 1 \in C([0, T]; L^1(\mathbb{R}^2)).
\]
Global existence

Here we assume that $f(u) = -u \left( e^{u^2} - 1 \right)$.

**Proposition**

Assume that $u_0 \in H^1(\mathbb{R}^2)$ and $u \in C([0, T); H^1(\mathbb{R}^2))$ solution to (4). Then

$$\|u(t)\|_{L^\infty} \leq \sqrt{2} \|u_0\|_{L^2}, \quad 0 < t < T.$$ 

We use the level set energy inequality for $c_k := M(1 - 2^{-k})$, where $M > 0$ will be chosen later. This idea was used by Caffarelli-Vasseur (*Annals of Mathematics (2010)*) for the quasi-geostrophic equation.
This leads to the following energy inequality for the level set function $u_k \overset{\text{def}}{=} (u - c_k)_+$:

$$\frac{d}{dt} \int_{\mathbb{R}^2} |u_k(t)|^2 \, dx + 2 \int_{\mathbb{R}^2} |\nabla u_k(t)|^2 \, dx \leq 0. \quad (7)$$

Let $t_0 > 0$, and denote by $T_k \overset{\text{def}}{=} t_0(1 - 2^{-k})$ and

$$U_k \overset{\text{def}}{=} \sup_{t \geq T_k} \left( \int_{\mathbb{R}^2} |u_k(t, x)|^2 \, dx \right) + 2 \int_{T_k}^{+\infty} \int_{\mathbb{R}^2} |\nabla u_k(t, x)|^2 \, dx \, dt.$$

By integrating (7) in time, and using some interpolation estimates, we obtain finally

$$U_k \leq A C^{k-1} U_{k-1}^{\frac{\alpha}{2}},$$

where

$$A \overset{\text{def}}{=} 2^{\frac{\alpha}{2}+4} M^{-\alpha} t_0^{-1} \|u_0\|_{L^2}^2,$$

and

$$C \overset{\text{def}}{=} 2^{\alpha+1}.$$
Taking $\alpha > 2$ and $M \equiv 2^{\frac{\alpha^2 + 10\alpha - 12}{2\alpha(\alpha - 2)}} t_0 \frac{1}{\alpha} \|u_0\|_{L^2}$, we get

$$\lim_{k \to \infty} U_k = 0.$$ 

The conclusion follows by letting $\alpha \to \infty$. 

Presented by: Mohamed Majdoub
Blowing-up solutions

**Claim:** If \( u_0 \in H^1 \setminus \{0\} \) with \( J(u_0) \leq 0 \), then \( T^* < \infty \).

The proof is quite standard. Set \( y(t) \overset{\text{def}}{=} \frac{1}{2} \int_0^t \| u(s) \|_{L^2}^2 \, ds \).

Since \( \left( uf(u) - 2F(u) \right) \geq \varepsilon F(u) \) for some \( \varepsilon > 0 \), we find that

\[
y(t)y''(t) \geq (1 + \eta) (y'(t))^2 \quad \text{for some} \quad \eta = \eta(\varepsilon) > 0.
\]

The fact that this ordinary differential inequality blows up in finite time ensures the claim.
Nonuniqueness

The proof is done in four steps. Set $\Omega \overset{\text{def}}{=} B(0, 1) \subset \mathbb{R}^2$.

**Step 1:** We construct a singular solution $Q$ to

$$
\begin{cases}
-\Delta Q = f(Q) & \text{in } \Omega, \\
Q = 0 & \text{on } \partial\Omega, \\
Q(0) = \infty, & \text{and } Q > 0.
\end{cases}
$$
We look for singular solutions which are radially symmetric. Letting $|x| \overset{\text{def}}{=} r = e^{-t}$, $y(t) = Q(e^{-t})$, we obtain
\[
\begin{cases}
-y''(t) = e^{-2t} f(y(t)), & t \geq 0 \\
y(0) = 0, & y(\infty) = \infty.
\end{cases}
\]

To solve the above problem, we proceed as follows.

- For any $\alpha > 0$, we consider the Cauchy problem

\[
(\mathcal{P}_\alpha) \begin{cases}
-y''(t) = e^{-2t} f(y(t)), & t \geq 0 \\
y(0) = 0, & y'(0) = \alpha,
\end{cases}
\]

and the associated elliptic problem

\[
(\mathcal{E}_\alpha) \begin{cases}
-\Delta u_\alpha = f(u_\alpha), & 0 < r < 1 \\
u_\alpha(1) = 0, & u_\alpha(0) = y_\alpha(\infty).
\end{cases}
\]

- Elliptic regularity + Existence and uniqueness of classical radial solution $\implies \exists ! \alpha_0$ s.t. $\lim_{t \to \infty} y_{\alpha_0}(t) = \ell \in (0, \infty)$. 

Presented by: Mohamed Majdoub
Set

\[
T(\alpha) = \sup \left\{ s \geq 0 ; \ y_\alpha > 0 \text{ on } (0, s) \right\}
\]

\[
I = \left\{ \alpha > 0 ; \ T(\alpha) < \infty \right\}
\]

\[
J = \left\{ \alpha > 0 ; \ \alpha \notin I \right\}
\]

The end of the proof of existence consists on showing that \( J \) is a non trivial interval.
Step 2: We prove that

- $-\Delta Q = f(Q)$ in $\mathcal{D}'(\Omega)$.

- $Q \in \mathcal{L}(\Omega)$.

- $\lim_{r \to 0} \|Q\|_{\mathcal{L}(|x|<r)} = 0$. 
The above classical solution in $\Omega \setminus \{0\}$ is extendable to a distribution solution in $\Omega$ thanks to the following technical lemma.

**Lemma**

Let $u \in C^2(\Omega \setminus \{0\})$, $u \geq 0$, such that $-\Delta u = f(u)$ in $\Omega \setminus \{0\}$. Then

1. $f(u) \in L^1(\Omega)$.
2. If $(-\log r)^\alpha u^q \in L^1(\Omega)$ for some $\frac{\alpha}{q-1} > 0$, then

   $$\Delta u + f(u) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega).$$

This lemma can be seen as an extension to dimension two of Ni-Sacks’ result.
Step 3: We take $Q$ as an initial data

\[
\begin{aligned}
&\begin{cases}
  u_t - \Delta u = f(u) \quad \text{in} \quad \Omega \\
  u(0) = Q, \quad u(|x| = 1) = 0
\end{cases}
\end{aligned}
\]  

We prove that (8) has a local solution $u \in L^\infty(L(\Omega))$.

Since the potential $e^{u^2} - 1$ is better than $L^1$, we can apply Brezis-Cazenave's result about regularization effect. This leads to $u \in L^\infty(L^\infty(\Omega))$. 

Presented by: Mohamed Majdoub

Energy critical heat equation ...
Step 4: Now we are able to conclude.

\[ u_0 = Q \quad \longrightarrow \quad v(t, x) = Q(x) \]

\[ u_0 = Q \quad \longrightarrow \quad u(t, x) \in L^\infty(L^\infty(\Omega)) \]

As \( Q(0) = \infty \), we find \( u \neq v \).
Concluding remarks
1. We believe that there is no existence in $H^s$ for $s < 1$. 
   (Work in progress)

2. The blow-up analysis can be refined for positive initial energy. 
   (Work in progress)
Thank You