ALMOST PERIODIC FUNCTIONS ASSOCIATED WITH A SINGULAR DIFFERENTIAL OPERATOR ON $(0,\infty)$

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ABSTRACT. We give a characterisation of almost periodic functions, associated with a class of singular differential operators L on $(0,\infty)$ including Bessel and Jacobi operators, as a closure in the sense of uniform convergence of finite sum of the form $\sum_k a_k|c(\lambda_k)|^2\phi_{\lambda_k}(x)$ where ϕ_{λ_k} are special eigenfunctions of L and $c(\lambda_k)$is the Jost-function associated with L.

1. INTRODUCTION

Many of the ideas and methods of classical Fourier analysis on the real line have been interesting and fruitful when studied in other contexts where at least some of the useful structures from the classical case persist.

A particular case is the harmonic analysis associated with a Sturm-Liouville operator. In that case, the underlying tool has been the generalized translation operators [C3] which have proved useful in specifying the class of Sturm-Liouville operators suitable for carrying out the analogy with classical harmonic analysis.

Here we will direct our attention to generalized almost periodic (g.a.p.) functions associated with Sturm-Liouville operators. The foregoing way of introducing the notion of g.a.p. functions goes back to Delsarte [D] and after him Levitan [Lev]. However, an important difference between the two papers strikes the attentive reader. While Levitan limits his concern to regular Sturm-Liouville operators and a characterisation of g.a.p. functions as a closure, in the sense of uniform convergence on the whole real line, of finite sum of the form $\sum a_k v(x,\lambda_k)$ where $v(x,\lambda_k)$ are special eigenfunctions of the Sturm-Liouville operator in consideration, Delsarte deals with a special but singular differential operator, namely the Bessel operator and gives a more complete study of g.a.p. functions.

As our approach here is similar to that of Delsarte and in order to give some motivations, we start with a short dicussion of Delsarte's results.

The general solutions $u(x,t)=f(x+t)+g(x-t)$ of the wave equation $u_{xx}-u_{tt}=0$ set off the importance of the role of the motion group on the real line. In particular, if f and g are Bohr-almost periodic (B-a.p.) functions so is $t\mapsto u(x,t)$ uniformly with respect to x. This suggests the following generalisation: Let L be a linear second order differential operator and let u be the solution of

$$L_x u-\frac{\partial^2 u}{\partial t^2}=0,\quad u(x,0)=f(x),\quad \frac{\partial u}{\partial t}(x,0)=g(t)$$

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What are the conditions on the data f and g in order that $t \mapsto u(x,t)$ be a B-a.p. function uniformly with respect to x ? In this case we will say (with Delsarte) that f and g are L-almost periodic (L-a.p.) functions.

Delsarte has investigate this question in the particular case where L is the Bessel operator

$$Lv = v'' + \frac{2\alpha+1}{x}v', \quad x > 0$$

with $-\frac{1}{2} < \alpha < \frac{1}{2}$. The theory he built is in many settings analogous to Bohr's one. The role of the exponentials is played by the modified Bessel function

$$j_\alpha(\lambda x) = \frac{2^\alpha \Gamma(\alpha+1)}{(\lambda x)^\alpha} J_\alpha(\lambda x), \quad \lambda > 0$$

and the motion group is replaced by the generalized translation operators associated with the Bessel operator [D]. Moreover Delsarte obtains a compactness property, series expansions in terms of $(j_\alpha(\lambda_k x))$ and prehilbertian structure in the space of L-a.p. functions. A fundamental role is played by the Mehler integral representation of $j_\alpha(\lambda x)$ in terms of the cosine function [W], namely

$$j_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} x^{-2\alpha} \int_0^x (x^2-t^2)^{\alpha-\frac{1}{2}} \cos \lambda t dt$$

The transformation operator defined by this formula is in fact a transmutation operator which allows one to study the Bessel operator in terms of the simpler operator $\frac{d^2}{dx^2}$ and thus the L-a.p. functions in terms of even B-a.p. functions. Furthermore, one can derive the orthogonality property of $(j_\alpha(\lambda_k x))$, $\lambda_k > 0$ from that of $(\cos \lambda_k x)$. At this stage the Fourier Bessel transform plays a decisive role.

In this paper we will work with differential operators of the form

$$L(A)u = u'' + \frac{A'(x)}{A(x)}u'$$

where A will have properties modeled on $L(A)$ being the radial part of the Laplace Beltrami operator on a non compact riemannian symmetric space of rank one [H]. The purpose is two-fold. On one hand, we have the feeling that it throws a certain amount of light on the results obtained by Delsarte and makes clear what is general and what is special about the Bessel operator. On the other hand, it leads to set off two distinct classes of Differential operators, one including Bessel operator and the other one including Jacobi operator, for which the theory can be extended. The distinction between the two classes is related to the fact that the corresponding spectral measures have distinct behaviours at the origin $\lambda = 0$.

The balance of the article is organized as follows: The next section contains some needed properties of solutions of $L(A)u - \lambda^2 u = 0$. Of particular interest are integral representations which will be established by the use of Riemann's method. This enables us to derive precise asymptotic behaviour of the Harish Chandra c function (intimately related with the Jost function in scattering theory). In section 3 we define $L(A)$-a.p. functions and set down some of their simple properties. Section 4 is devoted to the proof of the compactness property and the series expansions of $L(A)$-a.p.functions. The prehibertian structure is provided in section 5.

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2. The differential operator $L(A)$

2.1 Definitions and notations. Let $L(A)$ be the differential operator defined by

$$L(A)u = u'' + \frac{A'(x)}{A(x)}u', \quad x > 0$$

where the function A satisfies the following assumptions

(C) *A Convexity property*: we assume that A is increasing on $(0, \infty)$, $A(0) = 0$ and A'/A is decreasing, we then denote by 2ρ its limit at infinity

$$2\rho = \lim_{x\to\infty} \frac{A'(x)}{A(x)}$$

This hypothesis ensures the positivity of the generalized translation operators associated with $L(A)$ [C_1].

(R_0) *Regularity in the neighbourhood of zero.* We assume that A is of the form

$$A(x) = x^{2\alpha+1}\Delta(x), \quad \alpha > -\frac{1}{2}$$

where Δ is an even-C^∞ function and $\Delta(x) > 0$ for $x > 0$. So without loss of generality we assume $\Delta(0) = 1$.

(R_∞) *Regularity in the neighbourhood of infinity.* Two kind of growth properties are made according as $\rho = 0$ or $\rho > 0$. More precisely we assume

$$\text{for } x \to \infty, \qquad \frac{A'(x)}{A(x)} = \begin{cases} 2\rho + D_\rho(x), & \text{if } \rho > 0 \\ \frac{2\alpha+1}{x} + D_0(x), & \text{if } \rho = 0. \end{cases}$$

where D_ρ and D_0 are assumed to satisfy, for any $x_0 > 0$

$$\int_{x_0}^{\infty} t|D_j(t)|\,dt < \infty, \quad \int_{x_0}^{\infty} t|D_j'(t)|\,dt < \infty, \qquad j = 0, j = \rho$$

In the case where $\rho > 0$ we will assume, without loss of generality, that $A(x) \sim e^{2\rho x}$ $(x \to \infty)$.

Typical examples are the Bessel and Jacobi operators for which we have respectively

$$A(x) = x^{2\alpha+1}, \quad \alpha > -\frac{1}{2}, \ \rho = 0$$
$$A(x) = 2^{2\rho}(\cosh x)^{2\beta+1}(\sinh x)^{2\alpha+1}, \quad \alpha \geq \beta > -\frac{1}{2}, \ \rho = (\alpha+\beta+1) > 0$$

We shall be concerned with the asymptotic behaviour of solutions of the following differential equation

$$L(A)u + (\lambda^2 + \rho^2)u = 0, \quad \lambda \in \mathbb{C} \tag{1}$$

Assumption (R_0) shows that $x = 0$ is a regular singular point for (1); furthermore (1) has exactly one solution (regular solution), which will be denoted ϕ_λ, satisfying

$$\phi_\lambda(0) = 1, \quad \phi'_\lambda(0) = 0$$

So, $(x, \lambda) \mapsto \phi_\lambda(x)$ is an even function on $\mathbb{R} \times \mathbb{C}$, C^∞ with respect to x and analytic with respect to λ.

In order to analyze the behaviour of the solutions of (1) near infinity, we observe that the substitution $w = \sqrt{A}u$ transforms (1) into one of the following equations

$$(2^0) \qquad \begin{cases} w'' - \left(\dfrac{\alpha^2 - 1/4}{x^2} + q_0(x)\right) w + \lambda^2 w = 0 \\ q_0(x) = \dfrac{1}{4} D_0^2(x) + \dfrac{1}{2} D_0'(x) + \dfrac{\alpha + 1/2}{x} D_0(x) \end{cases} \qquad \text{if } \rho = 0$$

and

$$(2^\rho) \qquad \begin{cases} w'' - q_\rho(x) w + \lambda^2 w = 0 \\ q_\rho(x) = \dfrac{1}{4} D_\rho^2(x) + \dfrac{1}{2} D_\rho'(x) + \rho D_\rho(x) \end{cases} \qquad \text{if } \rho > 0$$

Assumptions (R_0) - (R_∞) show that q_0 is an even continuous function on $\mathbb{R}$, while q_ρ behaves like $(\alpha^2 - 1/4)x^{-2}$ near $x = 0$ and for any $x_0 > 0$, we have

$$\int_{x_0}^{\infty} t|q_j(t)|\, dt < \infty \qquad j = 0, j = \rho$$

In the two cases, one can uses Langer's result [Lang] to prove the following

Proposition 2.1. *There exist two positive constants N_0 and N_1 such that*

(1) *For* $|\lambda x| \leq N_1$

$$(2.3) \qquad \begin{aligned} \sqrt{A(x)}\phi_\lambda &= x^{\alpha+\frac{1}{2}} \left[j_\alpha(\lambda x) + O(x^2) \right] \\ (\sqrt{A(x)}\phi_\lambda(x))' &= (x^{\alpha+\frac{1}{2}} j_\alpha(\lambda x))' + x^{\alpha+\frac{3}{2}} O(1) \end{aligned}$$

(2) *For* $|\lambda x| > N_1$

$$(2.4) \qquad \begin{aligned} \sqrt{A(x)}\phi_\lambda &= x^{\alpha+\frac{1}{2}} j_\alpha(\lambda x) + \lambda^{-\alpha-\frac{3}{2}} \left[e^{i\lambda x} O(1) + e^{-i\lambda x} O(1) \right] \\ (\sqrt{A(x)}\phi_\lambda(x))' &= (x^{\alpha+\frac{1}{2}} j_\alpha(\lambda x))' + \lambda^{-\alpha-\frac{1}{2}} \left[e^{i\lambda x} O(1) + e^{-i\lambda x} O(1) \right] \end{aligned}$$

(3) *There exists a constant N_2 such that for all $\lambda \in \mathbb{C}$ and $x \geq 0$*

$$(2.5) \qquad \sqrt{A(x)}\phi_\lambda(x) \leq N_2 e^{|\Im\lambda| x} \left(\frac{x}{1 + |\lambda| x} \right)^{\alpha+\frac{1}{2}} e^{\left(\int_0^x \frac{t|V(t)|}{1+|\lambda| t} dt \right)}$$

Proof. Use the method of variation of constants and the behaviour of Bessel functions. For more details one can see ([C$_2$] or [T]). □

On the other hand, for any $\lambda \neq 0$, equation (1) has unique solutions which will be denoted by $\Phi_{\pm\lambda}$ (scattering solutions), such that when $x \to \infty$

$$\sqrt{A}(x)\Phi_{\pm\lambda}(x) = e^{\pm i\lambda x}W_{\pm}(x,\lambda), \quad \text{and} \quad \begin{cases} W_{\pm}(x,\lambda) = 1 + o(x^{-1}) \\ W'_{\pm}(x,\lambda) = o(x^{-1}) \end{cases} \tag{2.6}$$

For $\lambda = 0$, the asymptotic behaviour is different according as $\rho = 0$ or not. More precisely, equation (2^ρ) has two linearly independent solutions, which will be denoted by $\sqrt{A}\Phi$ and $\sqrt{A}\Psi$, such that

$$\begin{cases} \sqrt{A}(x)\Phi(x) - 1 = o(x^{-1}), & \left(\sqrt{A}(x)\Phi(x)\right)' = o(x^{-1}) \\ \sqrt{A}(x)\Psi(x) = O(x), & \left(\sqrt{A}(x)\Psi(x)\right)' = O(1) \end{cases} \quad (x \to \infty) \tag{2.7}$$

whereas equation (2^0) has two linearly independent solutions, which we denote again by $\sqrt{A}\Phi$ and $\sqrt{A}\Psi$, whose behaviours for large x are given by

$$\sqrt{A}(x)\Phi(x) = x^{\alpha+\frac{1}{2}}W_1(x), \qquad \lim_{x\to\infty} W_1(x) = 1 \tag{2.7^1}$$

$$\left(\sqrt{A}(x)\Phi(x)\right)' = (\alpha + \frac{1}{2})x^{\alpha-\frac{1}{2}}Z_1(x), \qquad \lim_{x\to\infty} Z_1(x) = 1 \tag{2.7^2}$$

$$\sqrt{A}(x)\Psi(x) = \begin{cases} x^{-\alpha+\frac{1}{2}}W_2(x), & \lim\limits_{x\to\infty} W_2(x) = 1, \quad \text{if } \alpha \neq 0 \\ \sqrt{x}\ln x V_2(x), & \lim\limits_{x\to\infty} V_2(x) = 1, \quad \text{if } \alpha = 0 \end{cases} \tag{2.7^3}$$

$$\left(\sqrt{A}(x)\Psi(x)\right)' = \begin{cases} (-\alpha + \frac{1}{2})x^{-\alpha-\frac{1}{2}}Z_2(x), & \lim\limits_{x\to\infty} Z_2(x) = 1, \quad \text{if } \alpha \neq 0 \\ \dfrac{\ln x}{2\sqrt{x}}Z_3(x), & \lim\limits_{x\to\infty} Z_3(x) = 1, \quad \text{if } \alpha = 0 \end{cases} \tag{2.7^4}$$

The behaviour of the scattering solution Φ_λ for large $|\lambda|$ both on the real axis and in $\mathbb{C}^*$ is going to play an important role in the further development. For doing it, again we have to distinguish the cases $\rho > 0$ and $\rho = 0$.

Proposition 2.2. *If $\rho > 0$, then for every $x > 0$ the function $\lambda \mapsto \Phi_\lambda(x)$ is a holomorphic function in $\{\lambda \in \mathbb{C} \mid \Im m(\lambda) > 0\}$ and continuous for $\Im m(\lambda) \geq 0$. Furthermore we have, as $\mid \lambda x \mid$ goes to infinity,*

$$\begin{aligned} \sqrt{A(x)}\Phi_{\pm\lambda}(x) &= e^{\pm i\lambda x}\left[1 + O(\frac{1}{\lambda x})\right] \\ (\sqrt{A(x)}\Phi_{\pm\lambda}(x))' &= \pm i\lambda e^{\pm i\lambda x}\left[1 + O(\frac{1}{\lambda x})\right] \end{aligned} \tag{2.8}$$

Proof. Combining the equation (2^ρ) with the boundary condition (2.6), and using the method of the variation of constants, we obtain the Volterra integral equation

$$\sqrt{A}(x)\Phi_\lambda(x) = e^{i\lambda x} + \int_x^\infty \frac{\sin\lambda(t-x)}{\lambda} q_\rho(t)\sqrt{A}(t)\Phi_\lambda(t)\,dt \tag{2.9}$$

Using the bound

$$\left|\frac{\sin z}{z}\right| \le C\frac{|z|}{1+|z|}e^{|\Im z|}, \quad \forall z \in \mathbb{C}$$

where C is an appropriate constant, we see that the method of successive approximations can be performed to give the desired result. $\square$

In order to study the case $\rho = 0$, we introduce the normalized Hankel function

$$\mathcal{H}_\alpha^{(1)}(\lambda x) = \sqrt{\frac{\pi}{2}}e^{\frac{i\pi}{2}(\alpha+\frac{1}{2})}(\lambda x)^{\frac{1}{2}}H_\alpha^{(1)}(\lambda x)$$

$H_\alpha^{(1)}$ being the Hankel function of order α and of the first kind. It has the following behaviour near the origin and at infinity

$$\mathcal{H}_\alpha^{(1)}(z) \underset{|z|\to\infty}{\simeq} e^{iz}, \quad \mathcal{H}_\alpha^{(1)}(z) \underset{|z|\to 0}{\simeq} \begin{cases} \pi^{-\frac{1}{2}}\Gamma(\alpha)e^{\frac{i\pi}{2}(\alpha-\frac{1}{2})}(z/2)^{-\alpha+\frac{1}{2}}, & \text{if } \alpha > 0 \\ -\frac{2i}{\pi}\sqrt{z}\ln z, & \text{if } \alpha = 0 \end{cases}$$

where $\ln z$ is taken to be the branch on $\mathbb{C} - i\mathbb{R}_-$ [W]. On that point it is useful to observe that there is a constant C such that for any $z \neq 0$ and $\alpha > -(1/2)$

$$|\mathcal{H}_\alpha^{(1)}(z)| \le C\left(\frac{|z|}{1+|z|}\right)^{-|\alpha|+\frac{1}{2}} e^{-|\Im z|} \tag{2.10}$$

Proposition 2.3. *If $\rho = 0$, then*

(1) *for every $x > 0$ the function $\lambda \mapsto \lambda^{-\frac{1}{2}}a(\lambda)\Phi_\lambda(x)$ is a holomorphic function in $\{\lambda \in \mathbb{C} \mid \Im(\lambda) > 0\}$ and continuous for $\Im(\lambda) \ge 0$ where $a(\lambda)$ is defined by*

$$a(\lambda) = \begin{cases} \lambda^\alpha, & \text{if } \alpha \neq 0 \\ (\ln\lambda)^{-1}, & \text{if } \alpha = 0 \end{cases}$$

(2) *For each $\lambda \neq 0$ with $\Im\lambda \ge 0$, $x \mapsto x^{-\frac{1}{2}}a(x)\sqrt{A}(x)\Phi_\lambda(x)$ is a continuous function for $x \ge 0$.*

(3) *For $\lambda \in \mathbb{C}$, with $\Im\lambda \ge 0$, we have the bounds*

$$|\sqrt{A}(x)\Phi_\lambda(x)| \le C\left(\frac{|\lambda|x}{1+|\lambda|x}\right)^{-|\alpha|+\frac{1}{2}} e^{-\Im\lambda x} \tag{2.11}$$

$$|\sqrt{A}(x)\Phi_\lambda(x) - \mathcal{H}_\alpha^{(1)}(\lambda x)| \le C\left(\frac{|\lambda|x}{1+|\lambda|x}\right)^{-|\alpha|+\frac{1}{2}} e^{-\Im\lambda x}\int_x^\infty t|q_0(t)|\,dt \tag{2.12}$$

$$|\sqrt{A}(x)\Phi_\lambda(x) - \mathcal{H}_\alpha^{(1)}(\lambda x)| \le \left(\frac{|\lambda|x}{1+|\lambda|x}\right)^{-|\alpha|+\frac{1}{2}} \frac{e^{-\Im\lambda x}}{|\lambda|}\int_x^\infty |q_0(t)|\,dt \tag{2.13}$$

Proof. Combining the equation (2^0) with the boundary condition (2.6), and using the method of the variation of constants, we obtain the Volterra integral equation

$$\sqrt{A}(x)\Phi_\lambda(x) = \mathcal{H}_\alpha^{(1)}(\lambda x) - \int_x^\infty G_\lambda(x,t)q_0(t)\sqrt{A}(t)\Phi_\lambda(t)\,dt$$

where G_λ is given by

$$\frac{i}{2\lambda}\left(\mathcal{H}_\alpha^{(1)}(\lambda x)\mathcal{H}_\alpha^{(1)}(-\lambda t) - \mathcal{H}_\alpha^{(1)}(\lambda t)\mathcal{H}_\alpha^{(1)}(-\lambda x)\right)$$

From the inequality (2.10) we derive for $x \leq t$ the following bound

$$|G_\lambda(x,t)| \leq C|\lambda|^{-1}\left(\frac{|\lambda|t}{1+|\lambda|t}\right)^{\alpha+\frac{1}{2}}\left(\frac{|\lambda|x}{1+|\lambda|x}\right)^{-\alpha+\frac{1}{2}} e^{|Im\lambda|(x-t)}$$

where C is an appropriate constant: we see again that the method of successive approximations can be performed to give the desired results. □

Remark 2.4. It should be noted that the bound (2.13) gives us

$$\sqrt{A}(x)\Phi_\lambda(x) = e^{i\lambda x}\left[1 + O(\frac{1}{\lambda x})\right], \quad (|\lambda|x \to \infty) \tag{2.14}$$

So (2.8) holds in the two cases $\rho > 0$ and $\rho = 0$.

Φ_λ and $\Phi_{-\lambda}$ are two linearly independent solutions of (1), so ϕ_λ is a linear combination of both

$$\phi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}$$

Using (2.7) and the fact that the wronskian $[\Phi_\lambda, \Phi_{-\lambda}]$ is independent of the variable x, we have

$$[\Phi_\lambda, \Phi_{-\lambda}] = -2i\lambda, \quad \text{and then} \quad [\phi_\lambda, \Phi_{-\lambda}] = -2i\lambda c(\lambda).$$

From the above two propositions we derive the asymptotic behaviour of the c function

Corollary 2.5. *In the case $\rho = 0$ the function $\lambda \mapsto a(\lambda)\lambda^{\frac{1}{2}}c(\lambda)$ is continuous on $\{\lambda \in \mathbb{C} \mid \Im m(\lambda) \leq 0\}$ whereas in the case $\rho > 0$, $\lambda \mapsto \lambda c(\lambda)$ is continuous on $\{\lambda \in \mathbb{C} \mid \Im m(\lambda) \leq 0\}$.*
In both cases $c(\lambda)^{-1}$ behaves like $\lambda^{\alpha+\frac{1}{2}}$ as $|\lambda|$ goes to infinity.

Thus the most important difference between $\rho > 0$ and $\rho = 0$ is that the function c has distinct behaviours at 0. However it has the same behaviour at infinity in both cases, $\rho = 0$ and $\rho > 0$.

Remark 2.6. In the case $\rho > 0$ the function $\lambda \mapsto \lambda c(\lambda)$ vanishes at $\lambda = 0$ only if ϕ_0 and Φ_0 are linearly dependent. In this case the function c is continuous on $\mathbb{R}$ and the function $x \mapsto \sqrt{A(x)}\phi_0(x)$ is bounded on $[0,\infty[$. Hence it follows that there exists a constant M such that

$$\sqrt{A(x)}|\phi_\lambda(x)| \leq M, \quad \forall\lambda \in \mathbb{R}\,\forall x \in [0,\infty[\tag{2.15}$$

Otherwise there exists $\gamma \neq 0$ and a continuous function h on $\mathbb{R}$ such that

$$c(\lambda) = \frac{\gamma}{\lambda} + h(\lambda) \quad \text{and} \quad \sqrt{A(x)}\phi_0(x) = O(x),\ (x \to \infty) \tag{2.16}$$

Remark 2.7. The function c is known as the Harish Chandra c-function. Clearly it plays a very fundamental role for the operator $L(A)$; indeed $|c(\lambda)|^{-2}$ represents the density of the spectral measure of $L(A)$ realized as an unbounded self adjoint operator in the Hilbert space $L^2(A(x))$ of square integrable functions on $(0,\infty)$ with respect to $A(x)dx$ (see [C$_3$]).

2.2 Product formula and integral representations. It has been shown in [C$_1$] (see also [C$_3$]) that, under the convexity property (C) (which we have not used yet)

(1) *Product formula.* For every $x \neq 0$, $y \neq 0$ there exists a non negative even and continuous kernel $z \mapsto T(x,y,z)$ supported in $[|x-y|, x+y]$ whose integral with respect to the measure $A(z)dz$ is less than or equal to one and such that

$$\phi_\lambda(x)\phi_\lambda(y) = \int_{|x-y|}^{x+y} \phi_\lambda(z)T(x,y,z)A(z)dz,\ \forall\lambda \in \mathbb{C} \tag{2.17}$$

(2) *Mehler representation.* For every $x \neq 0$, there exists a non negative even kernel $z \mapsto M(x,z)$ supported in $[-x,x]$ such that

$$\int_0^x M(x,z)dz \leq 1 \quad \text{and} \quad \phi_\lambda(x) = \int_0^x \cos\lambda z M(x,z)dz,\ \forall\lambda \in \mathbb{C} \tag{2.18}$$

(2.5) shows that $x \mapsto \phi_\lambda(x)$ is bounded for every complexe λ such that $|\Im\lambda| < \rho$; then using (2.17) we see that

$$\begin{aligned} &|\phi_\lambda(x)| \leq 1,\ \forall\lambda \in \mathbb{C},\ |\Im m(\lambda)| \leq \rho \\ &|\phi_\lambda(x)| \leq |\phi_0(x)| \leq 1, \quad \forall\lambda \in \mathbb{R} \end{aligned} \tag{2.19}$$

In [C-F-H] one can find another integral representation of Sonine type for ϕ_λ but we do not discuss this question here. However, we are interested in an integral representation of Φ_λ. In the case $\rho > 0$, we can apply a convenient Marchenko's procedure [A–M] to derive the following

Proposition 2.8. *Assume that $\rho > 0$; then for every complex λ with $\Im m(\lambda \geq 0)$ the solution Φ_λ has the form*

$$\sqrt{A(x)}\Phi_\lambda(x) = e^{i\lambda x} + \int_x^\infty K_\rho(x,t)e^{i\lambda t}dt, \quad x > 0 \tag{2.20}$$

where the kernel $K_\rho(x,t)$, defined for $0 < x \leq t$, has continuous partial derivatives and satisfies

$$\begin{aligned} &|K_\rho(x,t)| \leq \frac{1}{2}\sigma_0(\frac{x+t}{2})e^{\sigma_1(x)}, \quad \int_x^\infty |K_\rho(x,t)|dt \leq \sigma_1(x)e^{\sigma_1(x)} \\ &\int_x^\infty |\frac{\partial K}{\partial t}(x,t)|dt \leq \frac{1}{2}\sigma_0(x) + \sigma_0^2(x)\sigma_1(x)e^{(x)} \end{aligned} \tag{2.21}$$

Incidentally, this yields a new proof of proposition 2.2 when $\rho > 0$. Unfortunately, we cannot apply Marchenko's method in the case where $\rho = 0$ because of the slow decay of the function q_0. In order to address this difficulty Riemann's method seems to be the most promising way, but we need more hypotheses than (R_∞); we refer to ([G–M]), for further details.

2.3 Orthogonality property of ϕ_λ. We are going to show that the regular solution ϕ_λ, satisfy for $\lambda > 0$, an orthogonality property analogous to that satisfyed by circular functions. More precisely, we have

Theorem 2.9. *For every $\lambda > 0$ and $\mu > 0$*

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \phi_\lambda(x)\phi_\mu(x)A(x)dx = \begin{cases} 2|c(\lambda)|^2, & \text{if } \lambda = \mu \\ 0, & \text{if } \lambda \neq \mu. \end{cases}$$

Proof. We have

$$\sqrt{A(x)}\phi_\lambda(x) = c(\lambda)\sqrt{A(x)}\Phi_\lambda(x) + c(-\lambda)\sqrt{A(x)}\Phi_{-\lambda}(x)$$

Thus, the proof is an immediate consequence of (2.8) or (2.14). □

Remark 2.10. The asymptotic behaviour of ϕ_λ and ϕ_0 show that in general

$$\frac{1}{R}\int_0^R \phi_\lambda(x)\phi_0(x)A(x)dx, \quad \lambda > 0$$

has no limit when $R \to \infty$, whilst the limit

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R \phi_0^2(x)A(x)dx$$

is equal to infinity. Indeed, this is the case for the Bessel operator ($A(x) = x^{2\alpha+1}$) or the Jacobi operator ($A(x) = \sinh^{2\alpha+1}(x)$).

3. L-ALMOST PERIODIC FUNCTIONS

3.1 The operators $\mathcal{M}$ and (T^s), $s \geq 0$. Let $\mathcal{C}_*$ be the space of even and continuous functions on $\mathbb{R}$ endowed with the uniform norm $\|\,.\,\|_\infty$. Having in mind the product formula (2.17) and the Mehler representation (2.18) for ϕ_λ, we define the operators T^s and $\mathcal{M}$ for every $f \in \mathcal{C}_*$ by

$$T^s f(x) = \int_{|x-s|}^{x+s} f(z)T(x,s,z)A(z)dz \quad \text{and} \quad T^0 = Id \tag{3.1}$$

$$\mathcal{M}_x[f(\tau)] \equiv \mathcal{M}(f)(x) = \begin{cases} \int_0^x f(\tau)M(x,\tau)d\tau, & \text{if } x > 0 \\ f(0), & \text{if } x = 0 \end{cases} \tag{3.2}$$

Proposition 3.1. *The operators $\mathcal{M}$ and (T^s), $s \geq 0$, are linear and continuous from $\mathcal{C}_*$ into itself and for any $f \in \mathcal{C}_*$*

$$\|\mathcal{M}f\|_\infty \leq \|f\|_\infty \quad \textit{and} \quad \|T^s f\|_\infty \leq \|f\|_\infty,\ s \geq 0 \tag{3.3}$$

Proof. That $\mathcal{M}f \in \mathcal{C}_*$ and $T^s f \in \mathcal{C}_*$ for every $f \in \mathcal{C}_*$ is a consequence of [D–L] (pp 62-64), see also [Tr]. The inequalities (3.3) are a consequence of (2.17) and (2.18). □

Remark 3.2. We point out that $\mathcal{M}$ intertwines the differential operators $L(A)+\rho^2$ and the second differentiation operator D^2, that is

$$(L(A)+\rho^2)\mathcal{M} = \mathcal{M}D^2 \tag{3.4}$$

On the other hand the product formula (2.17) for ϕ_λ can be connected to that of cosine function, via the operator $\mathcal{M}$, namely for every $\lambda > 0$

$$\phi_\lambda(x)\phi_\lambda(s) = \frac{1}{2}\mathcal{M}_x\{\mathcal{M}_s[\cos\lambda(\tau+\nu) + \cos\lambda(\tau-\nu)]\} \tag{3.5}$$

where $\mathcal{M}_s$ acts with respect to ν and $\mathcal{M}_x$ acts with respect to τ.

3.2 The space $\mathcal{B}_L$. Having in mind remark 2.10, we introduce the space $\mathcal{B}_0$ of even Bohr almost periodic functions F whose means are equal to 0

$$\lim_{R\to\infty}\int_0^R F(x)dx = 0$$

The Fourier exponents of such functions are thus strictly positive.

Let $\mathcal{B}_L$ be the image of $\mathcal{B}_0$ under the operator $\mathcal{M}$

$$f \in \mathcal{B}_L \iff \exists F_f \in \mathcal{B}_0 \mid f = \mathcal{M}(F_f)$$

Definition 3.3. A function f is said to be L-almost periodic (L-a.p.) if it belongs to $\mathcal{B}_L$.

Le $f \in \mathcal{C}_*$ be such that $f = \mathcal{M}(F_f)$ with $F_f \in \mathcal{C}_*$ and introduce the function

$$\Psi_f(x,t) = \frac{1}{2}\mathcal{M}_x[F_f(t+\tau) + F_f(t-\tau)] \tag{3.6}$$

the operator $\mathcal{M}$ acting on the variable τ.

Proposition 3.4. *The function F_f belongs to $\mathcal{B}_0$ if and only if the function $\Psi_f(x,.)$ does, uniformly with respect to x.*

Proof. It is clear that if $\Psi_f(x,.)$ belongs to $\mathcal{B}_0$, uniformly with respect to x, so does $F_f(t) = \Psi_f(0,t)$. Assume now that $F_f \in \mathcal{B}_0$. Since $\mathcal{M}$ is a bounded operator, the function $t \mapsto \Psi_f(x,t)$ is an even Bohr almost periodic function uniformly with respect to x; moreover, the relation

$$\frac{1}{R}\int_0^R \Psi_f(x,t)dt = \frac{1}{2}\int_0^x \left(\frac{1}{R}\int_0^R [F_f(t+\tau)+F_f(t-\tau)]dt\right) M(x,\tau)d\tau$$

shows that the mean of $\Psi_f(x,.)$ is equal to 0. □

Remark 3.5. Using (3.4) one can prove that Ψ_f is a distribution solution of the Cauchy problem

$$\left(L_x(A)+\rho^2\right)\Psi_f - \frac{\partial^2}{\partial t^2}\Psi_f = 0, \quad \Psi_f(x,0) = f(x)$$

So proposition 3.4 says that f belongs to $\mathcal{B}_L$ if and only if the distribution solution of the above Cauchy problem is B.a.p in the variable t and uniformly with respect to x, the mean of which is equal to 0.

Example 3.6. Every function p of the form

$$p(x) = \sum_{n=1}^{r} \beta_n \phi_{\lambda_n}(x), \quad \beta_n \in \mathbb{C}, \quad \lambda_n > 0 \tag{3.7}$$

is L-a.p. and we have $p = \mathcal{M}(P_p)$ where

$$P_p(x) = \sum_{n=1}^{r} \beta_n \cos(\lambda_n x)$$

Definition 3.7. A spherical polynomial is any function of the form (3.7).

Theorem 3.8. *Let f be a function belonging to $\mathcal{B}_L$. Then*

(1) *For every $\epsilon > 0$ there exists a spherical polynomial p such that $\|f - p\|_\infty \leq \epsilon$.*
(2) *For every $s \geq 0$, $T^s(f)$ belongs to $\mathcal{B}_L$ and*

$$T^s(f)(x) = \mathcal{M}_x[\Psi_f(s,\tau)] \tag{3.8}$$

Proof. (1) Since $f \in \mathcal{B}_L$, there exists $F_f \in \mathcal{B}_0$ such that $f = \mathcal{M}(F_f)$. Thus there exists a trigonometrical polynomial P without constant term such that

$$\|F_f - P\|_\infty \leq \epsilon$$

Let $p = \mathcal{M}(P)$; this is a spherical polynomial and we have

$$\|f - p\|_\infty \leq \|F_f - P\|_\infty \leq \epsilon$$

(2) Let $p = \mathcal{M}(P)$ be a spherical polynomial. Using (3.7) we get

$$T^s(p)(x) = \mathcal{M}_x[\Psi_p(s,\tau)]$$

Then assertion (1) and the boundedness of T^s and $\mathcal{M}$ show that

$$\forall f \in \mathcal{B}_L, \quad T^s(f)(x) = \mathcal{M}_x[\Psi_f(s,\tau)]$$

Assertion (2) is thus a consequence of proposition 3.4. □

4. Spherical Fourier series expansions and compacity property of L-a.p.functions

Spherical Fourier series expansions. Let f be an L-a.p. function; that is $f = \mathcal{M}(F_f)$ where $F_f \in \mathcal{B}_0$. Let (λ_n) be the Fourier exponents of F_f and (β_n) be its Fourier coefficients. We have the classical relations

$$\beta_n = \lim_{R\to\infty} \frac{2}{R} \int_0^R F_f(x)\cos(\lambda_n x)dx$$

$$\sum_{n=1}^{\infty} |\beta_n|^2 = \lim_{R\to\infty} \frac{2}{R} \int_0^R |F_f(x)|^2 dx$$

According to the usual conventions, $\sum \beta_n \cos \lambda_n x$ is called the Fourier series of F_f and we write

$$F_f \sim \sum \beta_n \cos \lambda_n x$$

Definition 4.1. We will say that (λ_n) and (β_n) are respectively spherical Fourier exponents and spherical Fourier coefficients of f and we write

$$f(x) \sim \sum \beta_n \phi_{\lambda_n}(x)$$

This definition is justified by the following

Theorem 4.2 (The mean-value theorem). *Assume $\rho > 0$ or $\rho = 0$ and $-\frac{1}{2} < \alpha < 1$. Let f be an L-a.p. function whose spherical Fourier exponents and spherical Fourier coefficients are respectively (λ_n) and (β_n). Then for every $\lambda > 0$*

$$\lim_{R\to\infty} \frac{2}{R}\int_0^R f(x)\phi_\lambda(x)A(x)dx = \begin{cases} 2\beta_n|c(\lambda_n)|^2, & \text{if } \lambda = \lambda_n \text{ for some } n \\ 0, & \text{if } \lambda \neq \lambda_n,\ \forall n \end{cases}$$

Proof. Since F_f is in $\mathcal{B}_0$, for every $\epsilon > 0$ there exists a trigonometric cosine polynomial, without constant term $P = \sum_{n=1}^N \gamma_n \cos(\lambda_n x)$ such that

$$\|F_f - P\|_\infty \leq \epsilon \tag{4.1}$$

Let $p = \mathcal{M}(P)$, using the orthogonality property of ϕ_λ (theorem 2.9), we find

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R p(x)\phi_{\lambda_q}(x)A(x)dx = 2\gamma_q|c(\lambda_q)|^2 \tag{4.2}$$

Furthermore

$$|\gamma_q - \beta_q|^2 \leq \sum_{q=1}^\infty |\gamma_q - \beta_q|^2 = \lim_{R\to\infty} \frac{2}{R}\int_0^R |F_f(x) - P(x)|^2 dx \leq 2\epsilon \tag{4.3}$$

where $\gamma_q = 0$ for $q > N$. On the other hand

$$\frac{1}{R}\int_0^R [f(x) - p(x)]\phi_{\lambda_q}(x)A(x)dx = \frac{1}{R}\int_0^R [F_f(t) - P(t)]\mathfrak{J}_q(R,t)dt$$

where

$$\mathfrak{J}_q(R,t) = \int_t^R M(x,t)\phi_{\lambda_q}(x)A(x)dx$$

So using Hölder inequality one gets

$$\left| \frac{1}{R}\int_0^R [f(x) - p(x)]\phi_{\lambda_q}(x)A(x)dx \right|^2 \leq \frac{1}{R}\int_0^R |F_f(t) - P(t)|^2 dt . \frac{1}{R}\int_0^R |\mathfrak{J}_q(R,t)|^2 dt$$

Assume for a moment that there exists $R_0 > 0$ and $K > 0$ such that

$$\frac{1}{R}\int_0^R |\mathfrak{J}_q(R,t)|^2 dt \leq K^2, \quad \forall\, R \geq R_0 \tag{4.4}$$

then, according to (4.1)-(4.4), one gets

$$\left| \frac{1}{R}\int_0^R f(x)\phi_{\lambda_q}(x)A(x)dx - 2\beta_q|c(\lambda_q)|^2 \right| \leq \epsilon(K+1+2\sqrt{2}|c(\lambda_q)|^2)$$

and theorem 4.1 is established. Thus it remains to prove (4.4) which will be provided in several lemmas.

We first remark that

$$\int_0^R \mathfrak{J}_q(R,t)\cos(\mu t)dt = \int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx$$

so, by using Fourier-Plancherel formula, we have

$$\frac{1}{R}\int_0^R |\mathfrak{J}_q(R,t)|^2 dt = \frac{2}{\pi R}\int_0^\infty \left|\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx\right|^2 d\mu$$

Thus we have to prove that the right hand side of this equality remains bounded as $R \to \infty$. The first step is the following lemma

Lemma 4.3. *For every $m > 0$, there is a constant M such that*

$$\frac{1}{R}\int_m^\infty \left|\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx\right|^2 d\mu \leq M \tag{4.5}$$

Proof. According to corollary 2.5, $|c(\mu)|^{-2} = 0(|\mu|^{2\alpha+1})$ as $|\lambda| \to \infty$. Then for every $m > 0$ there exists a constant $M > 0$ such that

$$\int_m^\infty \left|\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx\right|^2 d\mu \leq M\int_m^\infty \left|\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx\right|^2 \frac{d\mu}{|c(\mu)|^2}$$

By using the spectral theorem associated with $L(A)$ [C$_2$], we have

$$\int_0^\infty \left|\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx\right|^2 \frac{d\mu}{|c(\mu)|^2} = 2\pi\int_0^R \phi_{\lambda_q}^2(x)A(x)dx$$

therefore we derive

$$\frac{1}{R}\int_m^\infty \left|\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx\right|^2 d\mu \leq \frac{2\pi M}{R}\int_0^R \phi_{\lambda_q}^2(x)A(x)dx$$

By theorem 2.9, the right hand side of this inequality, and thus the left one, remains bounded as $R \to \infty$. This establishes (4.5). $\square$

Lemma 4.4. *For every $m > 0$, there is a constant M such that*

$$\frac{1}{R}\int_0^m \left|\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx\right|^2 d\mu \leq M \tag{4.6}$$

Proof. An integration by parts shows that

$$\mu^2\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)dx = \lambda_q^2\int_0^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)\,dx + [\phi_{\lambda_q},\phi_\mu](R)$$

where $[f,g](x) = A(x)\,(f'g - fg')\,(x)$ is the wronskian of f and g.

i) In the case $\rho > 0$, (2.6)-(2.7) show that there exists a constant M such that

$$|\,[\phi_{\lambda_q},\phi_\mu](x)\,| \leq M, \quad \forall x,\mu \in \mathbb{R}$$

so as R goes to infinity, the left hand side of (4.6) remains bounded if and only if the same holds for

$$\frac{1}{R}\int_0^m \left|\int_1^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)\,dx\right|^2 \mu^4 d\mu$$

But since $|c(\mu)|^{-2} \simeq \mu^2 \geq M'|\mu|^4, (|\mu| \to 0)$, it follows that there exists M such that

$$\begin{aligned}\frac{1}{R}\int_0^R |\phi_{\lambda_q}(x)|^2 A(x)\,dx &\geq \frac{1}{MR}\int_0^m \left|\int_1^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)\,dx\right|^2 |c(\mu)|^{-2}d\mu \\ &\geq \frac{1}{R}\int_0^m \left|\int_1^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)\,dx\right|^2 \mu^4 d\mu\end{aligned}$$

The left hand side is bounded by theorem (2.9).

ii) In the case $\rho = 0$, again (2.6) and (2.7^1)-(2.7^4) show that as R goes to infinity

$$\forall \mu \in \mathbb{R} \;\; |\,[\phi_{\lambda_q},\phi_\mu](R)\,| = \begin{cases} O(R^{\alpha-\frac{1}{2}}), & \text{if } \alpha \neq 0 \\ O(\ln R/R), & \text{if } \alpha = 0 \end{cases}$$

and thus

$$\frac{1}{R}\int_0^m |\,[\phi_{\lambda_q},\phi_\mu](R)\,|^2\, d\mu = \begin{cases} O(R^{2\alpha-2}), & \text{if } \alpha \neq 0 \\ O\left(\ln^2 R/R^2\right) \end{cases}$$

Since $-1/2 < \alpha < 1$, the right hand is bounded. So, as above, it suffices to prove that

$$\frac{1}{R}\int_0^m \left|\int_1^R \phi_{\lambda_q}(x)\phi_\mu(x)A(x)\,dx\right|^2 \mu^4\, d\mu$$

remains bounded as R goes to infinity. But in the case $\rho = 0$, we have $|c(\mu)|^{-2} \simeq \mu^{2\alpha+1} \geq M'|\mu^4|, (|\mu| \to 0)$; then we conclude as in the first case. This concludes the proof of lemma 4.4 and finishes the proof of theorem 4.2. $\square$

Proposition 4.5. *Le f be an L-a.p. function whose spherical Fourier exponents and spherical Fourier coefficients are respectively (λ_n) and (β_n). Then for every $s > 0$, the spherical Fourier exponents of $T^s(f)$ are (λ_n) and its spherical Fourier coefficients are $(\beta_n \phi_{\lambda_n}(s))$.*

Proof. Let $f \in \mathcal{B}_L$. By theorem 3.8 the function $T^s(f)$ is in $\mathcal{B}_L$ and

$$T^s(f)(x) = \mathcal{M}_x[\Psi_f(s, \tau)]$$

Using the dominated convergence theorem and formula (2.17), we obtain

$$\lim_{R\to\infty} \frac{2}{R} \int_0^R \Psi_f(s,t) \cos \lambda t \, dt = \begin{cases} \beta_n \phi_{\lambda_n}(s), & \text{if } \lambda = \lambda_n \text{ for some } n \\ 0, & \text{if } \lambda \neq \lambda_n, \ \forall n \end{cases}$$

which means that the Fourier exponents and the Fourier coefficients of $\Psi_f(s.)$ are respectively (λ_n) and $(\beta_n \phi_{\lambda_n}(s))$. This concludes the proof. $\square$

Compacity property. Let $f \in \mathcal{B}_L$

$$f(x) \sim \sum \beta_n \phi_{\lambda_n}(x)$$

Then we have

$$F_f(x) \sim \sum \beta_n \cos \lambda_n x \quad \text{and} \quad \Psi_f(s, x) \sim \sum \beta_n \phi_{\lambda_n}(s) \cos \lambda_n x$$

Let $\mathcal{S}(f)$ the set of functions $\Psi_f(s, .), s \geq 0$, and their limits in the sense of uniform convergence on $\mathbb{R}$.

Proposition 4.6. *The family $\{\Psi_f(s, .), s \geq 0\}$ constitutes a majorisable set.*

Proof. We have to show that the functions $\{\Psi_f(s, .), s \geq 0\}$ are equally Bohr a.p. and equally uniformly continuous on $\mathbb{R}$. This is a consequence of (3.6) and the fact that $\mathcal{M}$ is bounded by 1 (see (2.18)). $\square$

Proposition 4.7. *Let $\Psi \in \mathcal{B}_0$. Then Ψ belongs to $\mathcal{S}(f)$ if and only if its Fourier series has the following form*

$$\Psi(x) \sim \sum a_n \phi_{\lambda_n}(s_n^*) \cos \lambda_n x$$

where s_n^* is such that there exists a sequence (s_m) of strictly positive numbers satisfying

$$\lim_{m\to\infty} \phi_{\lambda_n}(s_m) = \phi_{\lambda_n}(s_n^*)$$

Proof. The proof is a mimic of Delsarte's (see p. 304 [D]). $\square$

Remark 4.8. The proposition means that in the set $\mathcal{S}(f)$, the formal convergence of Fourier series is sufficient to ensure the uniform convergence.

Proposition 4.9. *Let* (s_m) *a sequence of positive numbers. Then there exists a subsequence* $(\Psi_f(s'_m, .))$ *uniformly convergent on* $\overline{\mathbb{R}}$.

Proof. For each fixed integer q, the function ϕ_{λ_q} takes its values in a finite interval $[0, \nu_q]$. Thus for every $x \in \mathbb{R}$ there exists $x_q^* \in [0, \nu_q]$ such that

$$\phi_{\lambda_q}(x) = \phi_{\lambda_q}(x_q^*)$$

Consider now a sequence $(\Psi_f(s_m, .))$; to each s_m there corresponds $s_{mq}^* \in [0, \nu_q]$ such that

$$\phi_{\lambda_q}(s_m) = \phi_{\lambda_q}(s_{mq}^*)$$

The sequence (s_{mq}^*) contains a convergent subsequence $(s_{m_jq}^*)$; let s_q^* be its limit. Then we have

$$\phi_{\lambda_q}(s_q^*) = \lim_{m_j \to \infty} \phi_{\lambda_q}(s_{m_jq}^*) = \lim_{m_j \to \infty} \phi_{\lambda_q}(s_{m_j})$$

Appealing to proposition 4.7, we see that the formal series

$$\sum a_q \phi_{\lambda_q}(s_q^*) \cos \lambda_q x$$

defines a function in $\mathcal{B}_L$ which is the uniform limit of the sequence $(\Psi_f(s_m, .))$. □

Theorem 4.10. *Le* $f \in \mathcal{B}_L$ *and* (s_m) *a sequence of positive numbers. Then the sequence* $(T^{s_m} f)$ *contains a subsequence uniformly convergent on* $\overline{\mathbb{R}}$.

Proof. Tis is an immediate consequence of the previous proposition and the boundedness of the operator $\mathcal{M}$. □

Remark 4.11. This compacity property of the space $\mathcal{B}_L$ is a partial generalisation of Bochner's theorem which characterises the space $\mathcal{B}_0$.

5. The Hilbert Structure

In this section we assume also that $\rho > 0$ or $|\alpha| < 1/2$. Let f be a function in $\mathcal{B}_L$; that is $f = \mathcal{M}(F_f)$ where $F_f \in \mathcal{B}_0$. Let (λ_n) be its spherical Fourier exponents and (β_n) its spherical Fourier coefficients. We have

$$F_f(x) \sim \sum \beta_n \cos \lambda_n x \quad \text{and} \quad f(x) \sim \sum \beta_n \phi_{\lambda_n}(x)$$

The series $\sum |\beta_n|^2$ is convergent and we have

$$\sum |\beta_n|^2 = \lim_{R \to \infty} \frac{2}{R} \int_0^R |F_f(x)|^2 \, dx \tag{5.1}$$

but for the series $\sum |\beta_n|^2 |c(\lambda_n)|^2$ this is no longer true. It turns out that the integral

$$\frac{1}{R} \int_0^R |f(x)|^2 A(x) \, dx$$

has no limit when R goes to infinity. This leads in turn to the introduction of the following subspace $\mathfrak{H}_L$ of $\mathcal{B}_L$.

First let $\mathcal{B}_0'$ be the subspace of functions $F \in \mathcal{B}_0$ whose primitives $G = \int_0^x F(t)dt$ are odd and Bohr-almost periodic.

Definition 5.1. We denote by $\mathfrak{H}_L$ the subspace of functions $f \in \mathcal{B}_L$ such that F_f is in $\mathcal{B}'_0$. For such a function we denote by G_f the primitive of F_f satisfying $G_f(0) = 0$.

It should be observed that every spherical polynomial belongs to $\mathfrak{H}_L$. On the other hand, if f belongs to $\mathfrak{H}_L$ then so does $T^s f$ for every $s \geq 0$. Indeed, by (3.8) we have to prove that $\Psi_f(s,.)$ belongs to $\mathcal{B}'_0$. But

$$\int_0^x \Psi_f(s,t)\,dt = \frac{1}{2}\int_0^s [G_f(x+\tau) + G_f(x-\tau)]M(s,\tau)\,d\tau$$

The right hand side is an odd and Bohr-almost-periodic function since G_f is, so the same is true the left hand side.

Let $f \in \mathfrak{H}_L$ and

$$F_f(x) \sim \sum \beta_n \cos \lambda_n x$$

then

$$G_f(x) \sim \sum \beta_n \frac{\sin \lambda_n x}{\lambda_n}$$

and thus the two series

$$\sum |\beta_n|^2 \quad \text{and} \quad \sum \frac{|\beta_n|^2}{\lambda_n^2} \tag{5.2}$$

are convergent.

Proposition 5.2. *Let $f \in \mathfrak{H}_L$ whose spherical Fourier series expansion is given by $f(x) \sim \sum \beta_n \phi_{\lambda_n} x$. Then the series $\sum |\beta_n|^2 |c(\lambda_n)|^2$ is convergent.*

Proof. Let $\mathbb{N}_1$ and $\mathbb{N}_2$ be the subsets of $\mathbb{N}$ defined by

$$\mathbb{N}_1 = \{n \in \mathbb{N} \mid \lambda_n \leq 1\}, \quad \mathbb{N}_2 = \{n \in \mathbb{N} \mid \lambda_n > 1\}$$

Corollary (2.5) shows that $c(\lambda)$ is bounded for $\lambda > 1$, so the series

$$\sum_{n \in \mathbb{N}_2} |\beta_n|^2 |c(\lambda_n)|^2$$

is convergent since the first series in (5.2) is. On the other hand, by the same corollary we know that when $\lambda \to 0$, $c(\lambda) \simeq \lambda^{-1}$ if $\rho > 0$ and $c(\lambda) \simeq \lambda^{-(\alpha+\frac{1}{2})}$ if $\rho = 0$. Therefore in the two cases, the convergence of the second series in (5.2) implies that of

$$\sum_{n \in \mathbb{N}_1} |\beta_n|^2 |c(\lambda_n)|^2$$

which proves the proposition. $\square$

Remark 5.3. For every $\lambda > 0$, the function $x \to \sqrt{A}\phi_\lambda$ is bounded on $[0,\infty[$. Hence if p is a spherical polynomial (see 3.7) then $\sqrt{A}p$ is bounded on $[0,\infty[$. We shall prove this boundedness for every $f \in \mathfrak{H}_L$.

Theorem 5.4. *For every $f \in \mathfrak{H}_L$ there exists a constant $M > 0$ such that*

$$\max_{x \geq 0} |\sqrt{A(x)} f(x)| \leq M[\|F_f\|_\infty + \|G_f\|_\infty] \tag{5.3}$$

Proof. Let $f \in \mathfrak{H}_L$; according to Definition 5.2, we have

$$f(x) = \int_0^x F_f(t) M(x,t) dt, \quad \text{with} \quad F_f \in \mathcal{B}'_0$$

Using the Fourier-Plancherel Formula and the representation (2.18) we have

$$f(x) = \int_0^\infty \mathfrak{F}(F_f)(\lambda) \phi_\lambda(x)\, d\lambda \tag{5.4}$$

with

$$\mathfrak{F}(F_f)(\lambda) = \int_0^x F_f(t) \cos \lambda t\, dt \tag{5.5}$$

Integrating (5.5) by parts, one gets

$$\mathfrak{F}(F_f)(\lambda) = G_f(x) \cos \lambda x + \int_0^x G_f(t) \lambda \sin \lambda t\, dt \tag{5.6}$$

There exist non-increasing functions G_f^+ and G_f^-, bounded by $\|G_f\|_\infty$, such that $G_f = G_f^+ - G_f^-$. Inserting this decomposition in (5.6) and using the second mean theorem, one gets

$$\|\mathfrak{F}(F_f)\|_\infty \leq M \|G_f\|_\infty \tag{5.7}$$

Now we write (5.4) in the form

$$\begin{aligned} \sqrt{A(x)} f(x) &= \int_0^1 \mathfrak{F}(F_f)(\lambda) \sqrt{A(x)} \phi_\lambda(x)\, d\lambda + \int_1^\infty \mathfrak{F}(F_f)(\lambda) \sqrt{A(x)} \phi_\lambda(x)\, d\lambda \\ &= I_0 + I_\infty \end{aligned}$$

From (2.4) we can assert that there exists a constant M such that

$$\forall x \geq 1, \forall \lambda \geq 1, \ |\sqrt{A(x)} \phi_\lambda(x)| \leq M \lambda^{-(\alpha + \frac{1}{2})}$$

On the other hand, there exist non-decreasing functions F_f^+ and F_f^-, bounded by $\|F_f\|_\infty$, such that $F_f = F_f^+ - F_f^-$. Then, using again the second mean theorem, we get

$$|\mathfrak{F}(F_f)(\lambda)| \leq 6 \|F_f\|_\infty \lambda^{-1}$$

so we deduce that

$$|I_\infty| \leq 6M \|F_f\|_\infty \int_1^\infty \lambda^{-(\alpha + \frac{3}{2})}\, d\lambda = 6M \left(-\alpha + \frac{1}{2} \right)^{-1} \|F_f\|_\infty \tag{5.8}$$

In order to estimate I_0, we have to distinguish the case $\rho > 0$ from the case $\rho = 0$.

1) If $\rho > 0$, according to the remark 2.6 we may assume that

$$c(\lambda) = \frac{\gamma}{\lambda} + h(\lambda)$$

where $\gamma \in \mathbb{R}^*$ and h is a continuous function. Then using (2.20) we have

$$\sqrt{A(x)}\phi_\lambda(x) = 2ia\frac{\sin \lambda x}{\lambda} + 2ia\int_x^\infty K_\rho(x,t)\frac{\sin \lambda t}{\lambda}\, dt + \\ + b(\lambda)e^{i\lambda x} + b(-\lambda)e^{-i\lambda x} + \int_x^\infty K_\rho(x,t)[b(\lambda)e^{i\lambda x} + b(-\lambda)e^{-i\lambda x}]\, dt$$

Then an integration over $[0,1]$ and (2.21) give

$$|\int_0^1 \sqrt{A(x)}\phi_\lambda(x)\, d\lambda| \leq 2|a|| \int_0^x \frac{\sin \mu}{\mu} d\mu| + 2\sup_{|\lambda|\leq 1}|b(\lambda)|[1+\sigma_1(x)e^{\sigma_1(x)}] + \\ 2|a|\int_x^\infty |K_\rho(x,t)|| \left(\int_0^t \frac{\sin \mu}{\mu} d\mu\right) |dt$$

It follows that there exists a constant M such that

$$\forall x \geq 1,\ |\int_0^1 \sqrt{A(x)}\phi_\lambda(x)\, d\lambda| \leq M \tag{5.9}$$

Using, once more, the second mean theorem and (5.7) we see that there exists a constant M such that

$$|I_0| \leq M\|G_f\|_\infty \tag{5.10}$$

The theorem is then a consequence of (5.8) and (5.10).

2) If $\rho = 0$ then we have to assume $|\alpha| < 1/2$. Integrating (2.5) with respect to λ one gets

$$|\int_0^1 \sqrt{A(x)}\phi_\lambda(x)\, d\lambda| \leq N_2\left(-\alpha + \frac{1}{2}\right)^{-1} x^{\alpha - \frac{1}{2}}\left[(1+x)^{-\alpha+\frac{1}{2}} - 1\right] \\ \leq N_2(1 + 2^{-\alpha+\frac{1}{2}}), \quad \forall x \geq 1$$

Using (5.7) and the above inequality we deduce that

$$|I_0| \leq M'\|G_f\|_\infty \tag{5.11}$$

and the theorem is a consequence of (5.8) and (5.11). □

We are now in position to prove the main result of this section.

Theorem 5.5 (Parseval relation). *Let f and g be two functions in $\mathfrak{H}_L$, whose spherical Fourier series expansions are*

$$f(x) \sim \sum a_n \phi_{\lambda_n}(x) \quad \textit{and} \quad g(x) \sim \sum b_n \phi_{\nu_n}(x)$$

Then the following limit exists and is finite

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R f(x)\overline{g}(x)A(x)\,dx$$

It is equal to 0 if f and g have no common spherical Fourier exponents; otherwise it is equal to

$$\sum a_n \overline{b}_n |c(\lambda_n)|^2$$

where the summation is over their common spherical Fourier exponents. Moreover, Parseval's relation holds:

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R |f(x)|^2 A(x)\,dx = \sum_0^\infty |a_n|^2 |c(\lambda_n)|^2$$

Proof. 1) We asume that f and g have the same spherical Fourier exponents (λ_n). There is a sequence (P_n) of odd trigonometric polynomials which converges to G_f uniformly on $\overline{\mathbb{R}}$, and such that (P_n') converges to F_f uniformly on $\overline{\mathbb{R}}$. So for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow \|G_f - P_n\|_\infty \leq \epsilon \quad \text{and} \quad \|F_f - P_n'\|_\infty \leq \epsilon \tag{5.12}$$

We write

$$P_n'(x) = \sum_{j=1}^{N_n} \alpha_j^n \cos \lambda_j x$$

where λ_j are among the spherical Fourier exponents of f and we set $\alpha_j^n = 0, \quad \forall j > N_n$. Bessel's inequality and proposition (5.2) show that the series $\sum_j a_j \overline{b}_j |c(\lambda_j)|^2$ is convergent, let S be its value. There is an integer n_1, which can be chosen greater than n_0, such that

$$\mid S - \sum_{j=1}^{N_{n_1}} a_j \overline{b}_j |c(\lambda_j)|^2 \mid \leq \epsilon \tag{5.13}$$

Using the triangle inequality we have

$$\begin{aligned} \mid \frac{1}{2R} \int_0^R f(x)\overline{g}(x)A(x)\,dx - S \mid \leq \mid & \frac{1}{2R} \int_0^R [f(x) - p_{n_1}(x)]\overline{g}(x)A(x)\,dx + \\ & \mid \frac{1}{2R} \int_0^R p_{n_1}(x)\overline{g}(x)A(x)dx - \sum_{j=1}^{N_{n_1}} \alpha_j^{n_1} \overline{b}_j |c(\lambda_j)|^2 \mid + \\ & \mid \sum_{j=1}^{N_{n_1}} \alpha_j^{n_1} \overline{b}_j |c(\lambda_j)|^2 - \sum_{j=1}^{N_{n_1}} a_j^{n_1} \overline{b}_j |c(\lambda_j)|^2 \mid + \\ & \mid \sum_{j=1}^{N_{n_1}} a_j^{n_1} \overline{b}_j |c(\lambda_j)|^2 - S \mid \end{aligned}$$

where we have set

$$p_{n_1}(x) = \sum_{j=1}^{N_{n_1}} \alpha_j^{n_1} \phi_{\lambda_j}(x)$$

i) Since f and g are in $\mathfrak{H}_L$, then appealing to (5.3) and (5.12) we see that there exists a constant M such that

$$| \frac{1}{R} \int_0^R [f(x) - p_{n_1}(x)]\overline{g}(x) A(x)\, dx | \leq M[\|F_g\|_\infty + \|G_g\|_\infty]\epsilon \tag{5.14}$$

ii) Using theorem 2.9 it is easy to see that

$$\lim_{R\to\infty} \frac{1}{2R} \int_0^R p_{n_1}(x)\overline{g}(x)A(x)dx = \sum_{j=1}^{N_{n_1}} \alpha_j^{n_1}\overline{b}_j |c(\lambda_j)|^2$$

Thus there is $R_0 > 0$ such that

$$| \frac{1}{2R} \int_0^R p_{n_1}(x)\overline{g}(x)A(x)dx - \sum_{j=1}^{N_{n_1}} \alpha_j^{n_1}\overline{b}_j |c(\lambda_j)|^2 | \leq \epsilon, \quad \forall R \geq R_0 \tag{5.15}$$

iii) We set $\Delta = \sum |b_j|^2 |c(\lambda_j|^2$. Using Bessel's inequality, we have that

$$| \sum_{j=1}^{N_{n_1}} \alpha_j^{n_1}\overline{b}_j |c(\lambda_j)|^2 - \sum_{j=1}^{N_{n_1}} a_j^{n_1}\overline{b}_j |c(\lambda_j)|^2 | \leq \Delta^{\frac{1}{2}} [\sum_{j=1}^{N_{n_1}} |a_j - \alpha_j^{n_1}|^2 |c(\lambda_j)|^2]^{\frac{1}{2}} \tag{5.16}$$

Now using the classical Parseval's formula and (5.12) we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} | \frac{a_j - \alpha_j^{n_1}}{\lambda_j} |^2 &= \lim_{R\to\infty} \frac{2}{R} \int_0^R |G_f(x) - P_{n_1}(x)|^2\, dx \leq 2\epsilon^2 \\ \sum_{j=1}^{\infty} |a_j - \alpha_j^{n_1}|^2 &= \lim_{R\to\infty} \frac{2}{R} \int_0^R |G_f(x) - P_{n_1}(x)|^2\, dx \leq 2\epsilon^2 \end{aligned} \tag{5.17}$$

Let $\mathbb{N}_1$ and $\mathbb{N}_2$ be as in the proof of proposition 5.2, corollary (2.5) shows that there is δ_1 and δ_2 such that

$$\forall j \in \mathbb{N}_1,\ |c(\lambda_j)|^2 \leq \frac{\delta_1}{\lambda_j^2}, \quad \text{and} \quad \forall j \in \mathbb{N}_2,\ |c(\lambda_j)|^2 \leq \delta_2$$

thus from (5.17) we deduce

$$\sum_{j\in\mathbb{N}_1} | a_j - \alpha_j^{n_1} |^2 |c(\lambda_j)|^2 \leq 2\delta_1\epsilon^2$$

$$\sum_{j\in\mathbb{N}_2} | a_j - \alpha_j^{n_1} |^2 |c(\lambda_j)|^2 \leq 2\delta_2\epsilon^2$$

Then it follows that

$$\sum_{j=1}^{N_{n_1}} |b_j| \mid a_j - \alpha_j^{n_1} \mid^2 \leq 2\sqrt{\Delta}\sqrt{(\delta_1+\delta_2)}\epsilon \tag{5.18}$$

From (5.13), (5.14), (5.15) and (5.18) we obtain for every $R > R_0$

$$\left| \frac{1}{2R}\int_0^R f(x)\overline{g}(x)A(x)\,dx - S \right| \leq \epsilon\left\{ M\left(\|F_g\|_\infty + \|G_g\|_\infty\right) + 2 + 2\sqrt{\Delta(\delta_1+\delta_2)}\right\}$$

2) In the case where f and g have no common spherical Fourier exponents, the proof is carried out by the same arguments with $S = 0$. □

Conclusion. For two functions f and g in $\mathfrak{H}_L$, we set

$$\|f\|_L^2 = \lim_{R\to\infty}\frac{1}{2R}\int_0^R |f(x)|^2 A(x)\,dx$$
$$(f \mid g) = \lim_{R\to\infty}\frac{1}{2R}\int_0^R f(x)\overline{g}(x)A(x)\,dx$$

Then the space $\mathfrak{H}_L$ equipped with the norm $\|.\|_L$ and the inner product $(|)$ is a prehilbertian space; the subspace of spherical polynomials is dense in $\mathfrak{H}_L$. Moreover, the spherical Fourier series expansion of $f \in \mathfrak{H}_L$ converges to f with respect to $\|.\|_L$.

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